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GIMRADA Research Note No. 6

INVESTIGATION OF THE GEOMETRICAL QUALITY OF THE RELATIVE AND ABSOLUTE ORIENTATION PROCEDURES AND THE FINAL RESULTS OF THE PHOTOGRAMMETRIC PROCEDURE

By K. Bertil P. Hallert

24 August 1962

U.S. ARMY
GEODESY, INTELLIGENCE AND MAPPING
RESEARCH AND DEVELOPMENT AGENCY
CORPS OF ENGINEERS

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U. S. ARMY ENGINEER
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SUMMARY

This paper covers a theoretical investigation of the determination of the basic geometrical quality of the relative and absolute orientation procedures and the propagation of the errors from these operations to the final results of the photogrammetric procedure, including the elements of the exterior orientation. The geometrical quality is determined as standard errors from the standard error of unit weight of the basic data: image coordinates, parallaxes, and model coordinates. The principle of compensation between the elements of the absolute and the relative orientations is applied. Comparisons of the geometrical quality to be expected from normal-angle, wide-angle, and superwide-angle photographs have been made.

It is concluded that:

a. The method of least squares and its laws of error propagation allow a well-defined theoretical determination of the geometrical quality to be expected in the final results of the photogrammetric procedure in terms of the standard errors of the y-parallaxes.

b. The theoretical geometrical quality of the elements of the exterior orientation after double- and single-point resections in space can be determined in this way only.

c. The influence of the errors of the interior orientation must be taken into account. A well-defined procedure for the calibration of the camera and photographs is therefore necessary. Also, here the method of least squares is of basic importance.

d. Tolerance limits for the relative and absolute orientations can be derived from the results of these investigations.

INVESTIGATION OF THE GEOMETRICAL QUALITY OF THE RELATIVE AND ABSOLUTE ORIENTATION PROCEDURES AND THE FINAL RESULTS OF THE PHOTOGRAMMETRIC PROCEDURE

I. INTRODUCTION

The photogrammetric procedure can be divided into four fundamental operations, viz.:

1. The photography, i. e., the creation of perspective images of the object to be measured.

2. The reconstruction of the ideal bundles of rays between the points of the object and the (exterior) perspective center of the camera lens at the moments of exposure.

3. The relative orientation of overlapping bundles of rays in order to make corresponding rays intersect and thereby create an optical model (real or imaginary).

4. The absolute orientation for scaling and orienting the model into a given coordinate system and the final coordinate determination, numerically or graphically.

None of the mentioned fundamental operations can be performed free from errors. The geometrical quality of the final results, therefore, is a function of the geometrical quality of the operations. For a rational investigation of the geometrical quality of the final results it is necessary to consider them separately.

In Research Note Nos. 1 and 2 on the basic geometrical quality of aerial photographs and on the plotting instruments, i. e., the fundamental operations 1 and 2, have been discussed. In this investigation, the fundamental operations 3 and 4 are considered and, in particular, the mutual relation between the operations (the principle of compensation) is studied. Finally, the geometrical quality of the end results is expressed in terms of the geometrical quality of the fundamental operations.

The investigations primarily refer to the practical operations in plotting instruments of first order but the results can be applied to all types of stereoscopic photogrammetric plotting instruments, and the same procedures can, with minor modifications, be used for the investigations of the geometrical quality of analytical photogrammetric methods of various kinds.

Important prerequisites for the following investigations of the theory of errors of the relative and absolute orientations are that

the geometrical qualities of the aerial photographs and of the plotting instruments have been carefully tested under operating conditions for the distinction between regular errors of various kinds and for the estimation of statistical values of the inevitable irregular errors. For such purposes, the method of least squares is the most reliable and convenient procedure, in particular, if regular errors can be obtained directly as parameters from the adjustment. From the standard error of unit weight the standard errors of the regular errors can be directly determined too, as well as of all functions of the basic observations. It is always desirable that the test procedures of aerial photographs and of plotting instruments be repeated several times and with certain intervals under varying outer conditions in order to see the possible variations of the regular errors themselves. From well-known statistical principles, tolerance limits for the regular errors can be created for the test whether the variations can be accepted.

In summary, it is assumed that the most important regular errors of the fundamental operations 1 and 2 (the photographs and the plotting instruments) have been determined and can be corrected for; and the irregular errors have been estimated as standard errors of unit weight. Furthermore, it is desirable that the statistical distribution of the residuals be tested and found to be normal on the 5-percent level, at least.

The problems considered here have partly been discussed in different connections by the author, and some results have been published in different languages (references 3, 4, and 5). Therefore, no complete derivations are shown where such treatment can be found in literature; instead, a summary of the most important formula systems is made for the purpose of facilitating the reading and derivation of the final formulas for the accuracy of the coordinates, the elevations, and the elements of the exterior orientation.

II. INVESTIGATION

1. <u>The Basic Geometrical Quality of the Relative and Absolute Orientations</u>. The purpose of the relative orientation is to make all pairs of corresponding rays in overlapping bundles intersect. According to projective geometry, this is in principle achieved when five pairs of rays have been brought to intersect, provided that some general conditions concerning the locations of the rays or the intersected points are fulfilled. In the plotting instruments, the relative orientation is usually first performed empirically through systematic changes of suitable elements of orientation in order to bring corresponding rays to intersect in at least five points. This is equivalent to the vanishing of y-parallaxes in the orientation points. There are many different methods in use for this preliminary

part of the relative orientation and all lead to the following situation in the plotting instrument: In the five primary orientation points the rays intersect more or less correctly but never exactly because the intersection has to be observed; i. e., the y-parallaxes have to be observed and corrected with suitable elements of orientation. This is a measuring operation which never can be made free from errors and therefore it is not possible to make the five pairs of rays intersect exactly. This is also clearly shown by the fact that even if the operator does not see any y-parallax in a certain detail of the model, there may be considerable y-parallaxes in details in the immediate area. Furthermore, in other points of the model than those five in which the y-parallaxes were corrected, and which are arbitrarily located, there are certainly considerable y-parallaxes found when the model is inspected. In practice, many operators try to correct such residual parallaxes through small changes of the elements of orientation, and the result is a kind of empirical adjustment of the relative orientation where the residual y-parallaxes are made as small as possible although usually only through subjective judgment. There is, under such circumstances, no defined, objectively determined principle used for the important question when the relative orientation is to be regarded as finished. In general, no measurements of the residual y-parallaxes are performed either, and therefore, it is not possible afterwards to make any statements concerning the geometrical quality of the relative orientation.

Here, it is always assumed that the residual y-parallaxes are measured in the orientation points or rather regions after the empirical procedure. It is further always assumed that the residuals are observed and measured in at least three details around each ideal point and that the average is used as the parallax. We assume such measurements to be made in 5, 6, 9, and 15 regularly located points or regions of the model. The principles of least squares are applied to the treatment of these residual y-parallaxes. As stated previously, it is assumed that the errors of the image coordinates and of the instrument in principle are of irregular character and normally distributed on the 5 percent level. Next, the basic differential formulas are summarized.

For the definitions and signs of the Wild Autographs, the following basic differential formulas have been derived for the relation between small changes of the elements of the exterior orientation of a projector and the changes of projected coordinates of approximately vertical photographs:

$$dx = dx_o - \frac{x}{h}\,dz_o - y\,d\kappa - \left(1 + \frac{x^2}{h^2}\right)h\,d\varphi + \frac{xy}{h}\,d\omega \qquad (1)$$

$$dy = -dy_0 - \frac{y}{h} dz_0 + x d\varkappa - \frac{xy}{h} d\varphi + \left(1 + \frac{y^2}{h^2}\right) h d\omega \qquad (2)$$

These formulas have been derived, e. g., in the author's textbook, reference 6, p. 292. When the elements of orientation take arbitrary values, the differential formulas of the general projective relations have to be used (reference 6, pp. 247-256 and reference 7). The following derivations are applicable to arbitrary conditions provided that the basic differential formulas are correctly derived in the manner indicated. Also, for iteration, the basic differential formulas are most important.

For a y-parallax, defined as a difference between two projected y-coordinates from two overlapping photographs 1 and 2 (left and right) as $p_y = y_1 - y_2$, the general differential formula (Fig. 1) is:

$$p_y = -dby_1 + dby_2 - \frac{y}{h} dbz_1 + \frac{y}{h} dbz_2 + x d\varkappa_1 - (x-b) d\varkappa_2 - \frac{xy}{h} d\varphi_1 + \\ + \frac{(x-b)y}{h} d\omega_2 + \left(1 + \frac{y^2}{h^2}\right) h d\omega_1 - \left(1 + \frac{y^2}{h^2}\right) h d\omega_2 \qquad (3)$$

Only five elements are to be determined through the relative orientation; therefore, the necessary differential formula for the following developments contains only five terms. Usually, only those terms which belong to projector 2 (the right one) are chosen, which means the dependent procedure of the relative orientation. If the independent procedure had been preferred, the elements $d\varkappa_1$, $d\varkappa_2$, $d\varphi_1$, $d\varphi_2$, and $d\omega_1$ or $d\omega_2$ would have been chosen. For the derivation given here, the end results must become identical for both procedures of the relative orientation.

For the dependent procedure, the following differential formula is:

$$p_y = dby_2 + \frac{y}{h} dbz_2 - (x-b) d\varkappa_2 + \frac{(x-b)y}{h} d\varphi_2 - \left(1 + \frac{y^2}{h^2}\right) h d\omega_2 \qquad (4)$$

The relation between the elevations of the model and the elements of orientation according to the dependent procedure is given by the following differential formula (reference 6, p. 294):

$$dh = \frac{hy}{b} d\varkappa_2 - \left(1 - \frac{x}{b}\right) dbz_2 + \frac{H^2 + (x-b)^2}{b} d\varphi_2 - \frac{(x-b)y}{b} d\omega_2 \qquad (5)$$

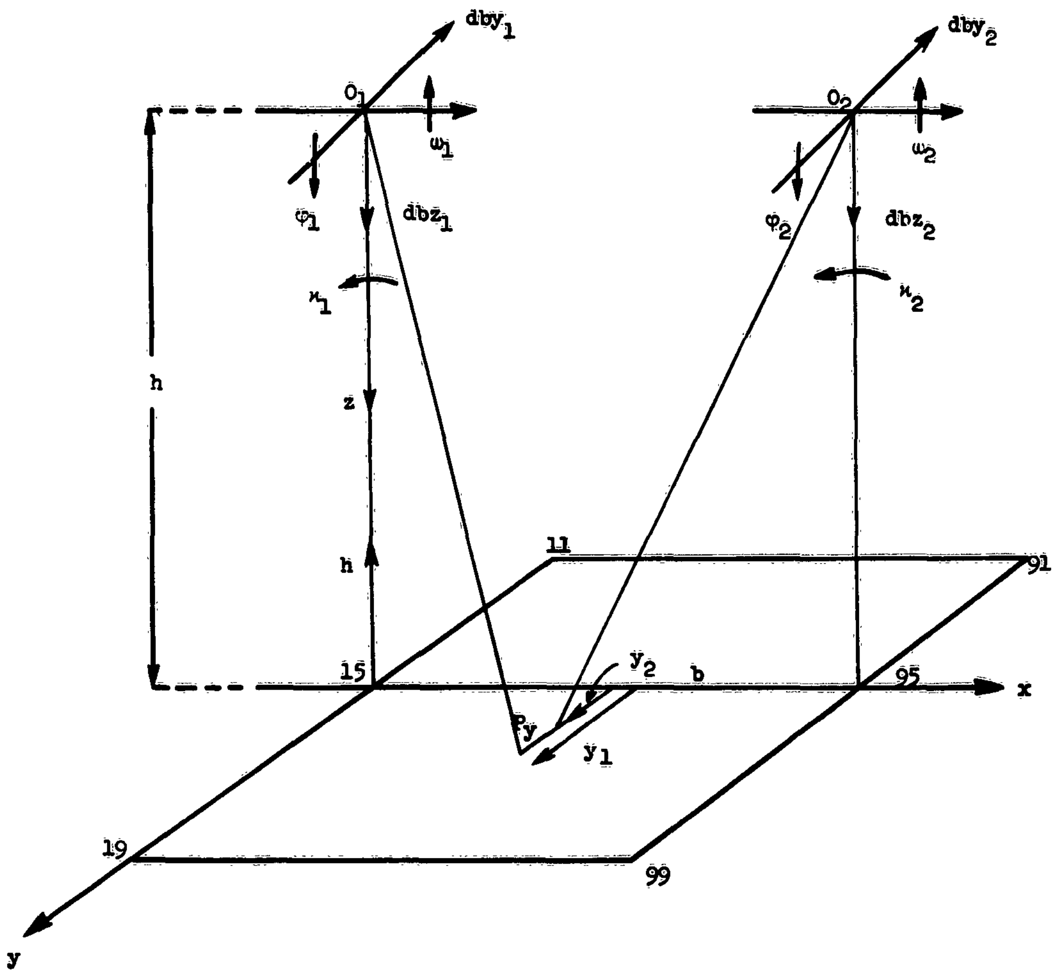

Fig. 1. Definition of the elements of the relative orientation. The arrows, defining the directions of the rotations, are placed between the actual axes and the reader.

The corresponding formulas for the x- and y-coordinates of the model (reference 6, p. 294) are:

$$dx = -\frac{xy}{b}\,d\kappa_2 - \frac{x(x-b)}{bh}\,dbz_2 - \frac{x\left\{(x-b)^2 + h^2\right\}}{bh}\,d\varphi_2 + \frac{x(x-b)y}{bh}\,d\omega_2 \quad (6)$$

$$dy = -\frac{dby_2}{2} + \left(\frac{x-b}{2} - \frac{y^2}{b}\right)d\kappa_2 - \left(\frac{x}{b} - \frac{1}{2}\right)\frac{y}{h}\,dbz_2 -$$

$$-\left\{\frac{(x-b)^2 + h^2}{b} + \frac{x-b}{2}\right\}\frac{y}{h}\,d\varphi_2 + \left\{\frac{y^2 + h^2}{2} + \frac{(x-b)y^2}{b}\right\}\frac{d\omega_2}{h} \quad (7)$$

These formulas are basic for the investigation of the error propagation from the relative orientation to the final elevations and coordinates of the model and also to the final elements of the exterior orientation. The principles can be described as follows. From at least five y-parallax observations the elements of the relative orientation can be determined with the aid of formula (4).

If redundant observations are available, an adjustment according to the method of least squares can be made, and the basic accuracy of the y-parallax measurements can be determined as standard error of unit weight. The determination becomes stronger with increasing number of redundant parallaxes. If the points in which the parallaxes are measured are located regularly, the normal equations become simple, and generalized solutions can be made. Simple forms can be used for the necessary calculations.

Corrections to preliminary measurements of elevations and coordinates x and y can then be computed from the determined corrections to the elements of the relative orientation and with the aid of formulas (5), (6), and (7). The accuracy (standard error) of these corrections can be computed from the accuracy (standard error of unit weight) of the basic observations from which the corrections of the elements of orientation were computed. However, in some points of the model, which are to be used for the absolute orientation, the errors emanating from the elements of the relative orientation and propagating through formulas (5), (6), and (7) become compensated by the elements of the absolute orientation. If there are no redundant control points (i. e., two points in planimetry and three points in elevation), the errors from the relative orientation can be completely compensated in the control points, and only the errors of the model coordinate measurements affect the control points after the absolute orientation. But the elements of the absolute orientation become affected with the errors from the relative orientation and in all other points of the model except the control points, the errors of the elements of the absolute orientation affect the elevations and coordinates in addition to the errors from the elements of the relative orientation. If redundant control points are available, some principle must be used for the distribution of the inevitable discrepancies in the control points in order to determine the elements of the absolute orientation uniquely. Also, here the principles of the method of least squares are applied. Through this procedure, which is founded upon the <u>principle of compensation</u>, the influence of the errors of the relative orientation upon the final elevations and coordinates of the model can be studied theoretically correctly, and the standard errors to be expected in the final data, including the elements of the exterior orientation, can be expressed in a well-defined and theoretically correct way. Evidently, the number of and the locations of the control points as well as the number of and locations of the model

points in which the y-parallaxes are observed and the basic standard error of unit weight must be taken into account. It also is shown from the formula systems (to be derived) how the geometrical data of the photographs (in particular, the opening angle and the overlap) influence the determination of the final accuracy.

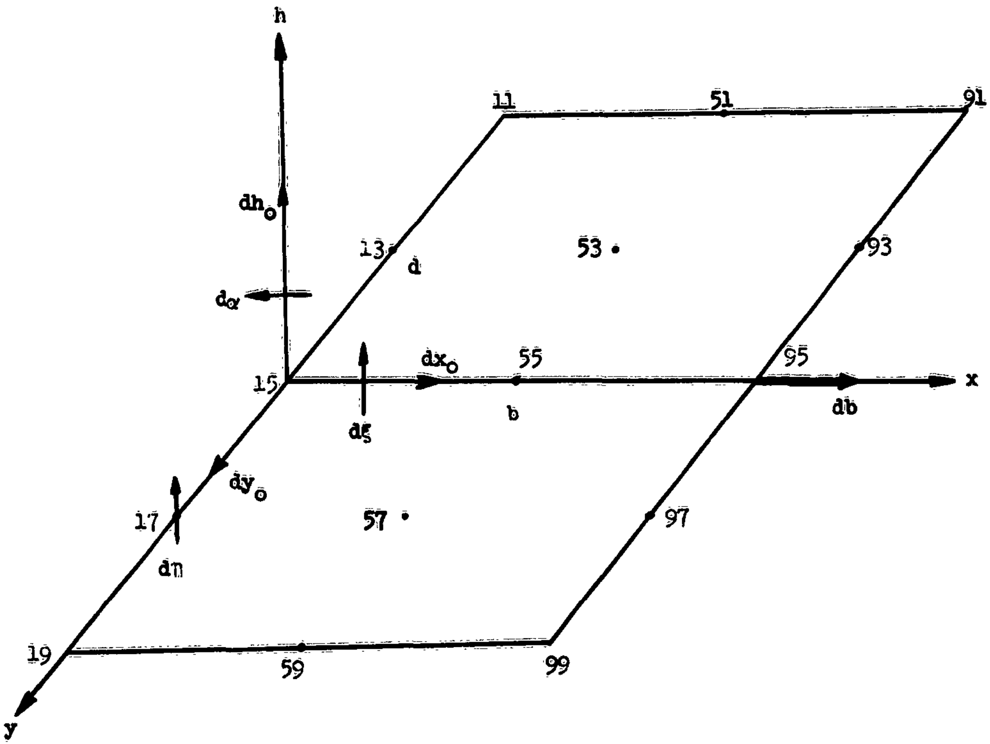

Fig. 2. Locations and notations of model points for the relative and absolute orientations and the differentials of the elements of the absolute orientation dx_o, dy_o, dh_o, $d\alpha$, $d\eta$, $d\xi$, and db.

Before the procedure is presented more in detail the basic formulas for the absolute orientation are shown (Fig. 2). For the elevations, the absolute orientation consists of one translation, dh_o, and two rotations, $d\eta$ and $d\xi$, around two horizontal axes, usually the y- and x-axes of the model coordinate system, respectively. The differential formula between the model elevations and the elements of the absolute orientation is:

$$dh = dh_o + xd\eta + yd\xi \qquad (8)$$

If at least three elevation discrepancies, dh_1, dh_2, and dh_3, have been observed in the model points, the corrections of the elements of the absolute orientation can be found from three equations of

type (8). If more than three elevation discrepancies have been measured, redundant data are available and a well-defined principle must be applied for unique determination of the corrections of the elements of the absolute orientation. For n elevation discrepancies $dh_1 \ldots dh_n$ in the model points $x_1y_1, \ldots x_ny_n$ the method of least squares gives unique solutions of the corrections. If the coordinates $x_1y_1 \ldots x_ny_n$ are translated into a system, the origin of which is the point of gravity, and these gravity coordinates are denoted $X_1Y_1 \ldots X_nY_n$, the following corrections (references 3 and 6, p. 315) are obtained:

$$dh_o = -\frac{[dh]}{n} \tag{9}$$

$$d\eta = \frac{[XY]\,[Ydh] - [YY]\,[Xdh]}{[XX]\,[YY] - [XY]^2} \tag{10}$$

$$d\xi = \frac{[XY]\,[Xdh] - [XX]\,[Ydh]}{[XX]\,[YY] - [XY]^2} \tag{11}$$

In planimetry, the absolute orientation consists of two translations dx_o, dy_o, one scale change $\frac{db}{b}$, and one rotation $d\alpha$ (Fig. 2): The basic differential formulas relating the differentials and the changes of the coordinates dx and dy follow:

$$dx = dx_o + x\frac{db}{b} - yd\alpha \tag{12}$$

$$dy = dy_o + y\frac{db}{b} + xd\alpha \tag{13}$$

The discrepancies in two points are evidently sufficient for the determination of the corrections of the preliminary values of the elements of the absolute orientation. If more than two points are available, the method of least squares is applied for the unique determination of the corrections. Again, if the coordinates x and y are translated to a coordinate system, the origin of which is located in the point of gravity, the corrections of the elements of the absolute orientation are obtained as follows from the discrepancies in n points (references 3 and 6, p. 316):

$$dx_o = -\frac{[dx]}{n} \tag{14}$$

$$dy_0 = -\frac{[dy]}{n} \qquad (15)$$

$$\frac{db}{b} = -\frac{[Xdx] + [Ydy]}{[XX] + [YY]} \qquad (16)$$

$$d\alpha = \frac{[Ydx] - [Xdy]}{[XX] + [YY]} \qquad (17)$$

2. Some Different Procedures of the Relative Orientation. As indicated previously, the relative orientation can be made with the aid of corrections or measurements of y-parallaxes in various numbers of model points and in various positions. Each such point combination constitutes certain conditions for the relative orientation and the geometrical quality of the operation and, in particular, the error propagation formulas become specific for each point combination. Some different point combinations are considered here, and then the error propagation and the principles of compensation are applied to one specific type of the relative orientation, the six-point combination, which is the most usual one in practice.

For the various point combinations the orientation points are chosen from such a scheme as is indicated in Fig. 2.

a. Five-Point Combination. As indicated previously, five points are generally a minimum for the relative orientation. As an example, the numerical determination of the corrections of the elements of orientation and the corresponding weight numbers and correlation numbers are shown here.

We assume the y-parallaxes after a preliminary orientation to be measured in the model points 15, 95, 19, 99, and 11. After formula (4), written as a correction formula, is applied to the five mentioned points and after the formula system is solved in some way, the following corrections of the elements of the relative orientation are obtained as direct functions of the measured y-parallaxes:

$$dby_2 = -p_{95} + \frac{h^2}{2d^2}(p_{11} + p_{19} - 2p_{15}) \qquad (18)$$

$$d\kappa_2 = \frac{p_{95} - p_{15}}{b} \qquad (19)$$

$$dbz_2 = \frac{h}{2d}(p_{11} + p_{19} + 2p_{95} - 2p_{15} - 2p_{99}) \qquad (20)$$

$$d\varphi_2 = \frac{h}{bd}\,(p_{19} + p_{95} - p_{15} - p_{99}) \tag{21}$$

$$d\omega_2 = \frac{h}{2d^2}\,(p_{11} + p_{19} - 2p_{15}) \tag{22}$$

The weight and correlation numbers can next be computed according to their definitions. The numbers are shown in the weight matrix in Table I.

Table I. Weight and Correlation Numbers of the Elements of the Relative Orientation; Five Orientation Points

	by_2	κ_2	bz_2	φ_2	ω_2
by_2	$\frac{3h^4+2d^4}{2d^4}$	$\frac{h^2-d^2}{bd^2}$	$\frac{h(3h^2-2d^2)}{2d^3}$	$\frac{h(3h^2-2d^2)}{2bd^3}$	$\frac{3h^3}{2d^4}$
κ_2		$\frac{2}{b^2}$	$\frac{2h}{bd}$	$\frac{2h}{b^2d}$	$\frac{h}{bd^2}$
bz_2			$\frac{7h^2}{2d^2}$	$\frac{7h^2}{2bd^2}$	$\frac{3h^2}{2d^3}$
φ_2				$\frac{4h^2}{b^2d^2}$	$\frac{3h^2}{2bd^3}$
ω_2					$\frac{3h^2}{2d^4}$

With the aid of the general law of error propagation and the terms from Table I the standard errors of all functions of the elements of the relative orientation, as for instance, computed from formulas (4) through (7), can be determined. The standard error of unit weight cannot be determined from measurements in five points only because no redundant measurements are available but must be determined from measurements in more points.

b. <u>Six-Point Combination</u>. This combination is the most common in the theory and practice of the relative orientation. The sixth point is a good check against gross errors in the parallax observations or corrections but means also a condition in which a discrepancy always is to be expected. Usually, the following model

points are used: 15, 95, 19, 99, 11, and 91. The general solution of the six-point orientation problem is well known (reference 6, p. 282), and the final formulas, only, are given here. For moderate elevation differences on the ground (h = const.), the following corrections are obtained:

$$dby_2 = \frac{1}{12d^2}\left\{- p_{15}(6h^2+4d^2) - p_{95}(6h^2+8d^2) + (p_{11}+p_{19})(3h^2+2d^2) + \right.$$
$$\left. + (p_{91}+p_{99})(3h^2-2d^2)\right\} \quad (23)$$

$$d\kappa_2 = \frac{1}{3b}(- p_{15} + p_{95} - p_{11} + p_{91} - p_{19} + p_{99}) \quad (24)$$

$$dbz_2 = \frac{h}{2d}(p_{91} - p_{99}) \quad (25)$$

$$d\varphi_2 = \frac{h}{2bd}(- p_{11} + p_{91} + p_{19} - p_{99}) \quad (26)$$

$$d\omega_2 = \frac{h}{4d^2}(- 2p_{15} - 2p_{95} + p_{11} + p_{91} + p_{19} + p_{99}) \quad (27)$$

$$[vv] = \frac{1}{12}(- 2p_{15} + 2p_{95} + p_{11} - p_{91} + p_{19} - p_{99})^2 \quad (28)$$

$$s_o = \sqrt{[vv]} \quad (29)$$

$$s_{s_o} = 0.70 s_o$$

The weight matrix is shown in Table II.

The error propagation from the basic y-parallax observations to arbitrary functions of the elements of the relative orientation can be studied with the aid of the laws of error propagation and the terms of Table II. It should be noted that the determination of the standard error of unit weight according to formulas (28) and (29) is extremely weak because there is only one redundant observation. Therefore, it is always advisable to use at least nine and preferably fifteen points for the determination of s_o. The formulas are given here. But the corrections of the elements of the

relative orientation can, in general, be computed from the formulas just given, provided that the residual y-parallaxes are small (less than about 0.05 mm in the photographs). The weight and correlation numbers are only slightly changed if more points are used.

Table II. Weight and Correlation Numbers of the Elements of the Relative Orientation; Six Orientation Points

	by_2	κ_2	bz_2	φ_2	ω_2
by_2	$\frac{2}{3} + \frac{h^2}{d^2} + \frac{3h^4}{4d^4}$	$-\frac{1}{3b}$	0	0	$\frac{3h^3 + 2d^2h}{4d^4}$
κ_2		$\frac{2}{3b^2}$	0	0	0
bz_2			$\frac{h^2}{2d^2}$	$\frac{h^2}{2bd^2}$	0
φ_2				$\frac{h^2}{b^2d^2}$	0
ω_2					$\frac{3h^2}{4d^4}$

c. Nine-Point Combination. This combination is of particular importance because there are four redundant observations. The standard error of unit weight, therefore, becomes determined with considerably higher reliability than in the six-point combination. The work is only slightly increased. The following formula systems refer to terrain with arbitrary elevation differences, and the previous formula systems, in principle, refer to flat terrain, although considerable elevation differences can be allowed if the preliminary orientation is performed so that the residual y-parallaxes are of the order of magnitude 0.05 mm in the photographs. It is difficult to derive theoretical tolerances for elevation differences and residual y-parallaxes. Practical tests, performed in connection with the international controlled experiments, within the International Society of Photogrammetry (ISP) 1956 through 1960, and reported in reference 5, have indicated that the results of adjustments of the relative orientation are not so sensitive to the elevation differences if the residual y-parallaxes are of the order of magnitude as indicated previously. The following formula systems have been derived by Ottoson and have been published in reference 5:

$$dby_2 = \frac{1}{18}(p_{15}-5p_{95}+p_{11}-5p_{91}+p_{19}-5p_{99}-2p_{51}-2p_{59}-2p_{55}) +$$

$$+ \frac{(2S-3A)}{18}d\omega_2 \qquad (30)$$

$$d\kappa_2 = \frac{1}{3b}(-p_{15} + p_{95} - p_{11} + p_{91} - p_{19} + p_{99}) + \frac{A}{3b}d\omega_2 \qquad (31)$$

$$dbz_2 = \frac{1}{12k}(-p_{11} + 5p_{91} + p_{19} - 5p_{99} + 2p_{51} - 2p_{59}) - \frac{B}{12k}d\omega_2 \qquad (32)$$

$$d\varphi_2 = \frac{1}{2bk}(-p_{11} + p_{91} + p_{19} - p_{99}) + \frac{C}{2bk}d\omega_2 \qquad (33)$$

$$d\omega_2 = \frac{6(qt + ru + sv) + ad + be + cf}{6(t^2 + u^2 + v^2) + d^2 + e^2 + f^2} \qquad (34)$$

The symbols are defined as follows:

$k = \left|\frac{y_i}{z_i}\right| \quad (L = 1 + k^2)$

z_i = the flying altitude over the actual point i and on the proper scale

y_i = the y-coordinate on the same scale

$$A = z_{15} - z_{95} + L(z_{11} - z_{91} + z_{19} - z_{99})$$

$$B = L(-z_{11} + 5z_{91} + z_{19} - 5z_{99} + 2z_{51} - 2z_{59})$$

$$C = L(z_{11} - z_{91} - z_{19} + z_{99})$$

$$S = z_{15} + z_{95} + z_{55} + L(z_{11} + z_{91} + z_{19} + z_{99} + z_{51} + z_{59})$$

$$a = -p_{15} - p_{95} - 2p_{11} - 2p_{91} + 4p_{51} + 2p_{59}$$

$$b = -p_{15} - p_{95} - 2p_{19} - 2p_{99} + 4p_{59} + 2p_{55}$$

$$c = -p_{11} - p_{91} + p_{19} + p_{99} + 2p_{51} - 2p_{59}$$

$$d = -z_{15} - z_{95} - 2Lz_{11} - 2Lz_{91} + 4Lz_{51} + 2z_{55}$$

$$e = -z_{15} - z_{95} - 2Lz_{19} - 2Lz_{99} + 4Lz_{59} + 2z_{55}$$

$$f = -Lz_{11} - Lz_{91} + Lz_{19} + Lz_{99} + 2Lz_{51} - 2Lz_{59}$$

$$q = -p_{51} - p_{59} + 2p_{55} \qquad t = -Lz_{51} - Lz_{59} + 2z_{55}$$

$$r = 2p_{15} - p_{11} - p_{19} \qquad u = 2z_{15} - Lz_{11} - Lz_{19}$$

$$s = 2p_{95} - p_{91} - p_{99} \qquad v = 2z_{95} - Lz_{91} - Lz_{99}$$

$$[vv]_9 = \frac{1}{36}\left\{6(q^2 + r^2 + s^2) + a^2 + b^2 + c^2 - \frac{\{6(qt + ru + sv) + ad + be + cf\}^2}{6(t^2 + u^2 + v^2) + d^2 + e^2 + f^2}\right\} \tag{35}$$

$$s_o = \sqrt{\frac{[vv]}{2}} \qquad s_{s_o} = 0.35 s_o \tag{36}$$

Confidence limits are:

On the level 5 percent: $0.6s_o - 2.9s_o$

On the level 1 percent: $0.5s_o - 4.4s_o$

Most of the formulas given previously can be considerably simplified because, provided the residual y-parallaxes are smaller than 0.05 mm in the photographs, the flying altitude can be chosen as an average for the model and, consequently, all z_i be substituted by h. Under this assumption the following formulas are obtained:

$$dby_2 = \frac{1}{6d^2}\left\{-p_{15}(d^2+2h^2) - p_{95}(3d^2+2h^2) + (p_{11}+p_{19}-2p_{55})(d^2+h^2) - (p_{91}+p_{99})(d^2-h^2) + (p_{51}+p_{59})h^2\right\} \tag{37}$$

$$d\kappa_2 = \frac{1}{3b}(-p_{15} + p_{95} - p_{11} + p_{91} - p_{19} + p_{99}) \tag{38}$$

$$dbz_2 = \frac{h}{12d}(-p_{11} + 5p_{91} + p_{19} - 5p_{99} + 2p_{51} - 2p_{59}) \tag{39}$$

$$d\varphi_2 = \frac{h}{2bd}\,(-\,p_{11} + p_{91} + p_{19} - p_{99}) \tag{40}$$

$$d\omega_2 = \frac{h}{6d^2}\,(-2p_{15}-2p_{95}+p_{11}+p_{91}+p_{19}+p_{99}+p_{51}+p_{59}-2p_{55}) \tag{41}$$

$$[vv] = \frac{1}{12}\,(-2p_{15}+2p_{95}+p_{11}-p_{91}+p_{19}-p_{99})^2 + \\ + \frac{1}{6}\left\{(p_{11}+p_{91}-2p_{51})^2+(p_{15}+p_{95}-2p_{55})^2+(p_{19}+p_{99}-2p_{59})^2\right\} \tag{42}$$

$$s_o = \sqrt{\frac{[vv]}{2}} \tag{43}$$

For the practical computations special forms can be used (Tables III and IV). The computations of the weight and correlation numbers can be made according to their definitions. For formulas

Table III. Form for the Computation of Corrections to the Elements of the Relative Orientation from Measured y-Parallaxes; Nine Points, Vertical Photographs A7

Point	p microns	I		II		III		IV		V		S	
		k	p · k	k	p · k	k	p · k	k	p · k	k	p · k	k	p · k
11		+1		+1		-1		-1		-1		-1	
51				+1				+2				+3	
91		-1		+1		+1		+5		+1		+7	
15		-1		-2		-1						-4	
55		-2		-2								-4	
95		-3		-2		+1						-4	
19		+1		+1		-1		+1		+1		+3	
59				+1				-2				-1	
99		-1		+1		+1		-5		-1		-5	
		+											
		-											
		Σ											

b = mm b^2 = mm^2

d = mm d^2 = mm^2

h = mm h^2 = mm^2

s_o = mm

$$db y_2 = \frac{1}{6}\cdot I + \frac{1}{6}\cdot\frac{h^2}{d^2}\cdot II = \qquad \sqrt{Q_{by_2by_2}} = \sqrt{0{,}5+\frac{h^4}{2d^4}+\frac{2h^2}{3d^2}} = \qquad s_{dby_2} = s_o\cdot\sqrt{Q_{by_2by_2}} =$$

$$d\kappa_2 = \frac{1}{3b}\cdot III\cdot\rho = \qquad \sqrt{Q_{\kappa_2\kappa_2}} = 0{,}817\cdot\frac{1}{b} = \qquad s_{d\kappa_2} = s_o\cdot\sqrt{Q_{\kappa_2\kappa_2}}\cdot\rho =$$

$$dbz_2 = \frac{h}{12d}\cdot IV = \qquad \sqrt{Q_{bz_2bz_2}} = 0{,}646\cdot\frac{h}{d} = \qquad s_{dbz_2} = s_o\cdot\sqrt{Q_{bz_2bz_2}} =$$

$$d\varphi_2 = \frac{h}{2bd}\cdot V\cdot\rho = \qquad \sqrt{Q_{\varphi_2\varphi_2}} = \frac{h}{bd} = \qquad s_{d\varphi_2} = s_o\cdot\sqrt{Q_{\varphi_2\varphi_2}}\cdot\rho =$$

$$d\omega_2 = \frac{h}{6d^2}\cdot II\cdot\rho = \qquad \sqrt{Q_{\omega_2\omega_2}} = 0{,}707\cdot\frac{h}{d^2} = \qquad s_{d\omega_2} = s_o\cdot\sqrt{Q_{\omega_2\omega_2}}\cdot\rho =$$

Table IV. Computation of the Accuracy of y-Parallax Measurements

Model:
Instrument:
Measured by:
Date:

Point	Measured y-parallax p (model)	Conditions I		II		III		IV		V		VI		VII		VIII		IX		X		Check	
		k	p.k	k	p.k	k	p.k	k	p.k	k	p.k	k	p.k	k	p.k	k	p.k	k	p.k	k	p.k	k	p.k
11		+1		+1										-1		+2		+1		+1		+5	
51				-2										-1				+1				-2	
91		-1		+1										-1		-2		+1		-1		-3	
13										+1				+4		-3		-2		-2		-2	
53										-2				+4				-2					
93										+1				+4		+3		-2		+2		+8	
15		-2				+1								-6		+2						-5	
55						-2								-6								-8	
95		+2				+1								-6		-2						-5	
17												+1		+4		-3		+2		+2		+6	
57												-2		+4				+2				+4	
97												+1		+4		+3		+2		-2		+8	
19		+1						+1						-1		+2		-1		-1		+1	
59								-2						-1				-1				-4	
99		-1						+1						-1		-2		-1		+1		-3	
$[p]_{15}$ =		+		+		+		+		+		+		+		+		+		+		+	
$[pp]_9$ =		-		-		-		-		-		-		-		-		-		-		-	
	Σ																					= Σ	
	Σ^2																						
	Σ^2	.35		.70		.70		.70		.70		.70		.2		.7		.14		.21		$= \frac{}{420} =$	

$$[vv]_9 = \frac{I + II + III + IV}{420} =$$

$$s_{09} = \frac{1}{2}\sqrt{[vv]_9} =$$

$$s_{m9} = \frac{1}{3}\sqrt{[pp]_9} =$$

$$s_{015} = \sqrt{\frac{[vv]_{15}}{10}} =$$

$$s_{m15} = \sqrt{\frac{[pp]_{15}}{15}} =$$

Enlargement image:model= 1:

On the image scale:

s_{015} = mm
s_{m15} = mm
s_{09} =
s_{m9} = mm

$= \quad = [vv]_{15}$

Computed by:
Checked by:

(30) through (34) the numbers become complicated. For most practical purposes, however, it is sufficient to use the weight and correlation numbers from formulas (37) through (41). These are shown in the matrix, Table V.

Table V. Weight and Correlation Numbers of the Elements of the Relative Orientation; Nine Orientation Points

	by_2	κ_2	bz_2	φ_2	ω_2
by_2	$\frac{1}{2} + \frac{2h^2}{3d^2} + \frac{h^4}{2d^4}$	$-\frac{1}{3b}$	0	0	$\frac{3h^3 + 2d^2h}{6d^4}$
κ_2		$\frac{2}{3b^2}$	0	0	0
bz_2			$\frac{5h^2}{12d^2}$	$\frac{h^2}{2bd^2}$	0
φ_2				$\frac{h^2}{b^2d^2}$	0
ω_2					$\frac{h^2}{2d^4}$

A comparison between Tables II and V shows that there are only minor changes of the weight and correlation numbers. This means that the accuracy of the corrections of the elements of the relative orientation is only slightly increased if y-parallaxes from nine points are used instead of those from six points. But the reliability of the standard error of unit weight is considerably increased from the six-point to the nine-point adjustment. Further increase of the reliability can be obtained from an adjustment of measurements in 15 points. Therefore, the formulas for the computation of the standard error of unit weight from 15 points are shown here (but without derivation) because this has been done earlier (reference 5).

d. Fifteen-Point Combination. A complete adjustment can be made from y-parallax observations in 15 regularly located points but only the formulas for the standard error of unit weight are given here. We have:

$$
\begin{aligned}
[vv] = {} & \frac{1}{12}\left(-2p_{15}+2p_{95}+p_{11}-p_{91}+p_{19}-p_{99}\right)^2 + \\
& + \frac{1}{6}\Big\{(p_{11}+p_{91}-2p_{51})^2+(p_{15}+p_{95}-2p_{55})^2+(p_{19}+p_{99}-2p_{59})^2 + \\
& + (p_{13}+p_{93}-2p_{53})^2+(p_{17}+p_{97}-2p_{57})^2\Big\} + \frac{1}{210}\left(-p_{11}+4p_{13}-\right. \\
& - 6p_{15}+4p_{17}-p_{19}-p_{51}+4p_{53}-6p_{55}+4p_{57}-p_{59}-p_{91}+4p_{93}-6p_{95}+4p_{97}- \\
& \left. - p_{99}\right)^2 + \frac{1}{60}\left(2p_{11}-3p_{13}+2p_{15}-3p_{17}+2p_{19}-2p_{91}+3p_{93}-2p_{95}+\right. \\
& \left. + 3p_{97}-2p_{99}\right)^2 + \frac{1}{30}\left(p_{11}-2p_{13}+2p_{17}-p_{19}+p_{51}-2p_{53}+2p_{57}-p_{59}+\right. \\
& \left. + p_{91}-2p_{93}+2p_{97}-p_{99}\right)^2 + \frac{1}{20}\left(p_{11}-2p_{13}+2p_{17}-p_{19}-p_{91}+2p_{93}-\right. \\
& \left. - 2p_{97}+p_{99}\right)^2
\end{aligned} \tag{44}
$$

$$
s_o = \sqrt{\frac{[vv]}{10}} \tag{45}
$$

$$
s_{s_o} = 0.22s_o
$$

Confidence limits are:

On the level 5 percent: $0.7s_o - 1.8s_o$

On the level 1 percent: $0.6s_o - 2.2s_o$

It can be noted that the 4 first terms of formula (44) are identical with formula (42). Furthermore, each of the 10 parentheses of formula (44) contains a condition, and the standard error of unit weight is determined from the discrepancies in these conditions. The last condition of formula (44) is, in particular, the condition which shows effects of possible radial distortion. For the practical computation of formula (44) it is suitable to use forms. Such a form, which can be used for the treatment of formula (44) as well as of formula (42) is shown in Table IV.

e. <u>Rules for the Determination of y-Parallaxes in the Wild Autographs</u>. We assume the y-parallaxes in the model to be

measured with the by_1 (left) or by_2 (right) projector. The readings of the elements have been recorded in the form as well as the readings of all the elements of orientation before the measurements.

The readings of the by_1 and by_2 before the measurements are denoted by_{1_0} and by_{2_0}, respectively, and the results of the readings (averages) from the y-parallax measurements are denoted by_{1_m} and by_{2_m}, respectively. Distinction is made between the four combinations, base in, base out, by_1, and by_2, respectively. The y-parallax is denoted p_y.

Base in, measurements with by_1 (left):

$$+p_y = by_{1_m} - by_{1_0}$$

Base in, measurements with by_2 (right):

$$+p_y = by_{2_0} - by_{2_m}$$

Base out, measurements with by_1 (left):

$$+p_y = by_{1_0} - by_{1_m}$$

Base out, measurements with by_2 (right):

$$+p_y = by_{2_m} - by_{2_0}$$

If these rules are applied to the determination of the y-parallaxes the preceding formulas give the corrections of the elements of the relative orientation with the correct signs.

3. The Principle of Compensation and the Derivation of Formulas for the Standard Errors in Planimetry and Elevation from Single Models. For the investigation of the propagation of the errors of the relative orientation to the final results, the compensating effect of the absolute orientation and the consequences of this must be taken into account. We assume the relative orientation to be performed up to a certain standard so that no significant corrections to the elements of the relative orientation can be found from the measured y-parallaxes in the orientation points (regions). The quality of the relative orientation can then be expressed in terms of the standard error of unit weight of the y-parallax measurements. The coordinates and elevations of the model are not to be regarded as errorless because the elements of the relative orientation still are affected with errors resulting from the lack of

perfection of this operation as shown by the standard error of unit weight. The errors of the coordinates and elevations are usually described as caused by model deformations and are analytically shown by formulas (5) through (7). Also, the residual y-parallaxes in other model points than the orientation points are functions of the errors of the elements of the relative orientation according to formula (4).

Statistical expressions for the model errors dx, dy, dh and the residual y-parallaxes can be found from the mentioned formulas by applying the general law of error propagation. The weight and correlation numbers to be used depend upon the number of and locations of the orientation points (Tables I, II, and V) for 5, 6, and 9 orientation points, respectively. For the following examples, we use the terms of Table II because this example is in best agreement with practice.

As an example of the determination of the standard errors of the coordinates, the elevations, or the y-parallaxes of a model after the relative orientation but before the absolute orientation we use the elevations according to formula (5). In fact, dh refers to the flying altitude, and the standard error s_h to be determined first is the standard error of the determination of the flying altitude after the relative orientation, only. This is different from the elevation differences of the model after the absolute orientation.

Applying the general law of error propagation to formula (5) and noting from Table II that many of the correlation numbers are zero, we find:

$$Q_{hh} = \frac{h^2 y^2}{b^2} Q_{\varkappa_2\varkappa_2} + \left(1 - \frac{x}{b}\right)^2 Q_{bz_2bz_2} + \frac{\{h^2 + (x-b)^2\}^2}{b^2} Q_{\varphi_2\varphi_2} + \frac{(x-b)^2 y^2}{b^2} Q_{\omega_2\omega_2} - \frac{2(b-x)\{h^2 + (x-b)^2\}}{b^2} Q_{\varphi_2 bz_2} \tag{46}$$

After the weight and correlation numbers are substituted from Table II the following formula is obtained:

$$Q_{hh} = \frac{h^2}{b^2}\left\{\frac{2y^2}{3b^2} + \frac{(x-b)^2}{2d^2} + \frac{(h^2 + x^2 + b^2 - 2bx)^2}{b^2d^2} + \frac{3(x-b)^2y^2}{4d^4} + \frac{(x-b)(h^2 + x^2 + b^2 - 2bx)}{bd^2}\right\} \tag{47}$$

The standard error of the elevations (flying altitudes) is found from

$$s_h = s_o \sqrt{Q_{hh}} \tag{48}$$

where s_o is the standard error of unit weight of the y-parallaxes.

In order to show the order of magnitude of the standard errors for a certain procedure, formula (48) is plotted in Fig. 3 for wide-angle photographs and 60 percent overlap $\left(b = d = \frac{3h}{5}\right)$ and for $s_o = 1$.

Similarly, formulas (6) and (7) have been plotted. The results are shown in Figs. 4 and 5.

It must be emphasized that the standard errors shown in Figs. 3, 4, and 5 refer to the models after the relative orientation

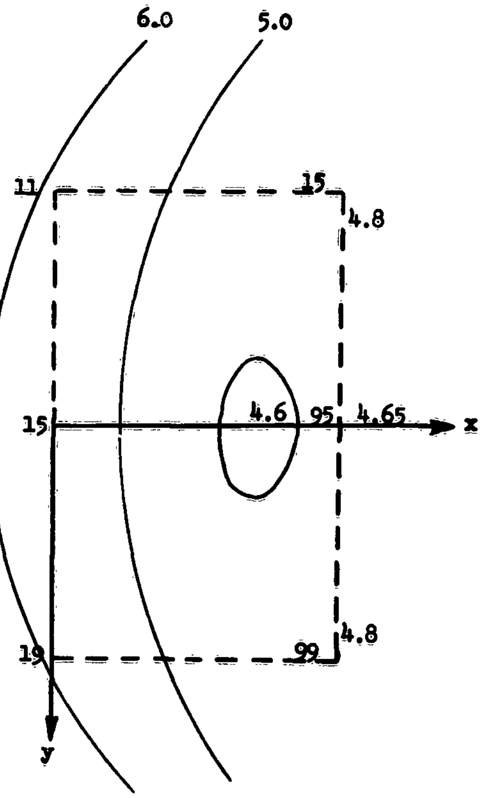

Fig. 3. Standard errors of the elevations of the model (from the air base) resulting from the relative orientation only. Vertical wide-angle photographs, about 60 percent overlap, dependent relative orientation, $s_o = 1$.

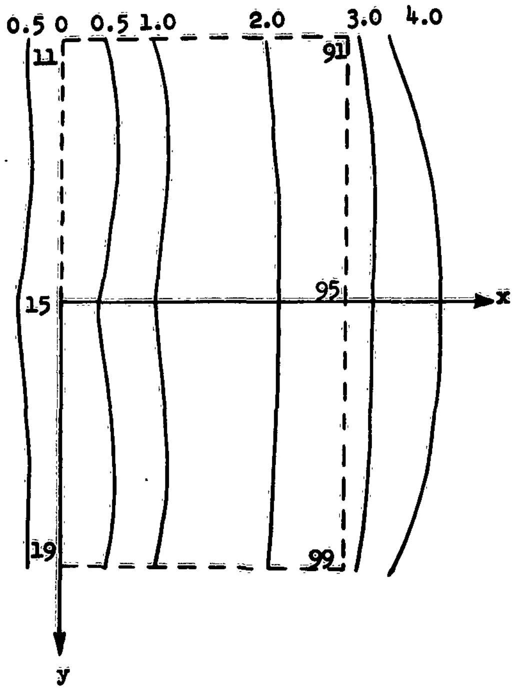

Fig. 4. Standard errors of the x-coordinates of a model resulting from the relative orientation only. Vertical wide-angle photographs, about 60 percent overlap, dependent relative orientation, $s_o = 1$.

only, and that other relations will be found after the absolute orientation because of the compensation effects. The standard errors as shown in the figures are of great interest too, however, because the error propagation in different procedures of aerial triangulation is based upon the errors of the individual models after the relative orientation. If the scale is transferred from model to model with the aid of elevation measurements in the vicinity of the nadir points of the models, the standard errors of the actual elevations can be found from formula (48) and the corresponding graphical plotting. It should be noted that the standard error of the elevation-measuring operation has to be taken into account in addition to the influence from the relative orientation. The standard error of the y-parallax corrections according to formula (4) is also of interest and can be computed concerning the elevations as has been shown previously. A graphical presentation of the standard error according to formula (4) including the measurement in arbitrary model points is shown in Fig. 6.

For the derivation of the error propagation formulas after the absolute orientation, the elevations and the coordinates x and y are treated separately.

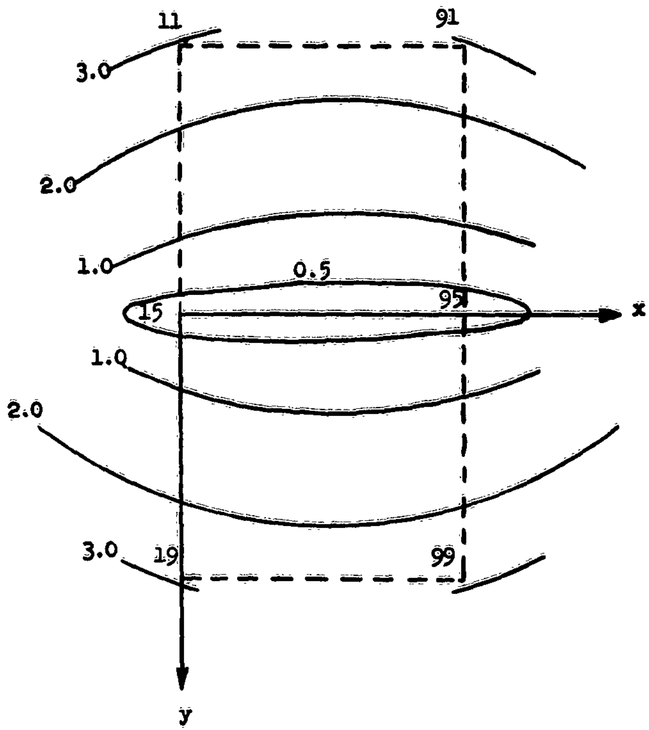

Fig. 5. Standard errors of the y-coordinates of a model resulting from the relative orientation only. Vertical wide-angle photographs, about 60 percent overlap, dependent relative orientation, $s_o = 1$.

a. The Elevations. We assume the relative orientation to be performed and adjusted according to the method of least squares after measurements of the y-parallaxes in the six points. For the absolute orientation, elevation control points are assumed to be located in the vicinity of the model points 11, 19, and 95 (Fig. 2). The model coordinates of these points are:

$$\begin{array}{lll} x_{11} = 0 & x_{19} = 0 & x_{95} = b \\ y_{11} = -d & y_{19} = d & y_{95} = 0 \end{array} \qquad (49)$$

The elevation discrepancies in the control points after a preliminary absolute orientation are defined as measured value minus given value. It must first be noted that the discrepancies consist of two parts, viz., one, caused by the elevation-measuring operation (the correction of the horizontal parallaxes) and then another caused by the model deformations resulting from the elements of the relative orientation and expressed by formula (5). These two parts of the elevation discrepancies are assumed to be independent of each other and are therefore considered separately.

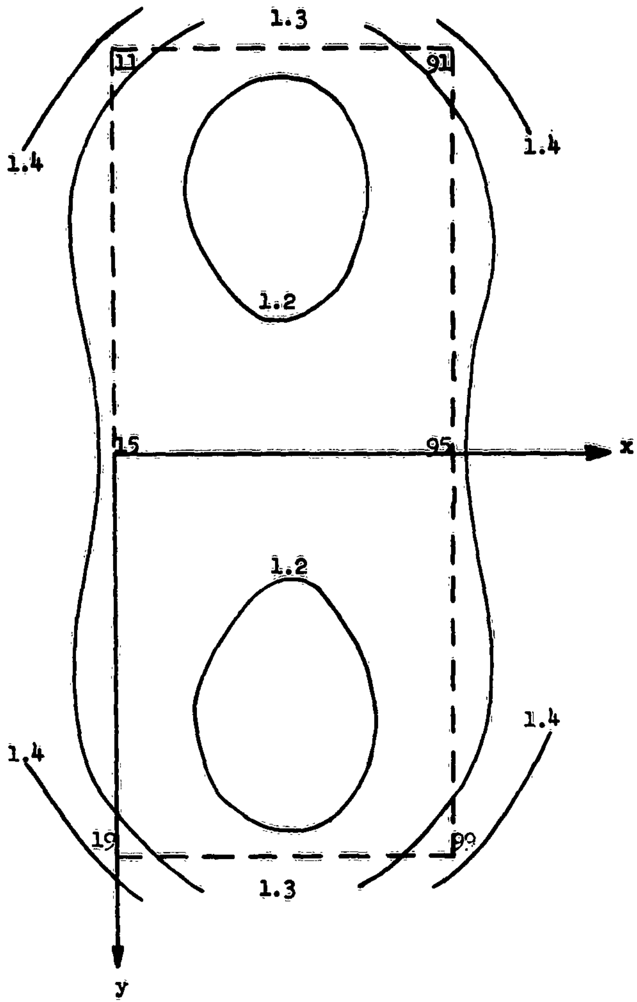

Fig. 6. Standard error of residual y-parallaxes after the relative orientation and including the measurements in arbitrary model points. Vertical wide-angle photographs, about 60 percent overlap. $s_o = 1$.

(1) The Error Propagation from the Elevation Measurements. The elevation measurements are assumed to be of equal quality. A certain weight variation of the elevation measurements can be derived from the found weight variation of the image coordinates (reference 1), but more determinations of the basic weights of the image coordinates are desired before the weighting of the measurements of the coordinate differences or parallaxes. Furthermore, the standard error of unit weight of the parallax measurements is determined as an adjusted value from measurements in at least 9 and preferably 15 points. Therefore, the weight relations also become adjusted to a certain extent. When sufficient determinations of the

weights of the basic image coordinates are available it might be of interest to introduce weights also in the measurements of differences of projected image coordinates. The influence of the elevation measurement errors in the control points upon the elements of the absolute orientation can be found from formula (8).

Applying this to the three elevation control points (49), we find:

$$dh_{11} = dh_o - dd\xi$$

$$dh_{19} = dh_o + dd\xi$$

$$dh_{95} = dh_o + db\eta \tag{50}$$

Hence,

$$dh_o = \frac{dh_{11} + dh_{19}}{2} \tag{51a}$$

$$d\eta = \frac{2dh_{95} - dh_{11} - dh_{19}}{2b} \tag{51b}$$

$$d\xi = \frac{dh_{19} - dh_{11}}{2d} \tag{51c}$$

The corresponding weight and correlation numbers are:

$$Q_{h_oh_o} = \frac{1}{2} \tag{52a}$$

$$Q_{\eta\eta} = \frac{3}{2b^2} \tag{52b}$$

$$Q_{\xi\xi} = \frac{1}{2d^2} \tag{52c}$$

$$Q_{h_o\eta} = -\frac{1}{2b} \tag{52d}$$

If the elevations of other model points than the control points are measured simultaneously, corrections to the measured data can be computed from the results of the corrections in the control points, i. e., from formula (8) in which the results in formulas (51) are substituted after the signs are changed. The weight number of a corrected elevation in an arbitrary point x,y can then be obtained

from formulas (8) and (52). The original elevation measurement in the actual point has the weight number 1. The standard error of the corrected elevation in the point x,y then becomes:

$$s_h = s_{oh} \sqrt{\frac{3}{2} + \frac{3x^2}{2b^2} + \frac{y^2}{2d^2} - \frac{x}{b}} \tag{53}$$

s_{oh} denotes the standard error of unit weight of the elevation measurement and is a function of the standard error of unit weight of the corresponding parallax correction which is assumed to be of the same magnitude as the correction or measurement of a y-parallax. Because the relation between parallaxes and elevation differences is

$$dh = \frac{h}{b}\,dp$$

the standard error of unit weight s_{oh} is found as

$$s_{oh} = \frac{h}{b}\,s_o \tag{54}$$

It is assumed that the same measuring technique is used for horizontal as well as for vertical or y-parallaxes (preferably stereoscopic measurements). Otherwise, the weight relation between monocular and stereoscopic measurements must be introduced, usually 1:2.

If different weights had been assigned to the measurements of the elevations as a result of the weight variations of the image coordinates and the reconstructed bundles of rays, such weights could have been introduced in formula (50). The further procedure is in principle not influenced as to whether weights are used.

If more points than the minimum number three are used for the absolute orientation, the inevitable discrepancies must be treated in some well-defined way. Again, if the least squares principle is applied, formulas (9) through (11) can be used. The expressions of the accuracy become comparatively simple if the control points are located symmetrically around the center point of the model. When five control points are assumed in the positions 55, 11, 91, 19, and 99, the standard error of an elevation as a result of the elevation measurements only, would become (reference 4):

$$s_h = s_{oh} \sqrt{\frac{6}{5} + \frac{(2x - b)^2}{4b^2} + \frac{y^2}{4d^2}} \tag{55}$$

The chosen point combination is of particular practical importance and is much used in connection with cadastral photogrammetric measurements.

(2) <u>The Compensation of the Model Deformations in the Control Points and the Error Propagation</u>. Each model point is affected with an elevation error resulting from the relative orientation according to formula (5). The errors in the control points can be found after substitution of the coordinates, x and y of the points, respectively. These errors will next be compensated by the elements of the absolute orientation and according to formula (8), into which the coordinates x and y are substituted. The compensation means that in each control point the sum of the influences from the relative and the absolute orientation procedures becomes zero. For the points 11, 19, and 95 is found:

$$\text{Point 11:} \quad -\frac{hd}{b}\,d\kappa_2 - dbz_2 + \frac{h^2+b^2}{b}\,d\varphi_2 - dd\omega_2 + dh_0' - dd\xi' = 0$$

$$\text{Point 19:} \quad \frac{hd}{b}\,d\kappa_2 - dbz_2 + \frac{h^2+b^2}{b}\,d\varphi_2 + dd\omega_2 + dh_0' + dd\xi' = 0$$

$$\text{Point 95:} \quad \frac{h^2}{b}\,d\varphi_2 + dh_0' + bd\eta' = 0 \tag{56}$$

The elements of the absolute orientation are denoted with a prime minus sign in order to point out the different meaning in comparison with formulas (51).

Hence,

$$dh_0' = dbz_2 - \frac{h^2+b^2}{b}\,d\varphi_2 \tag{57a}$$

$$d\eta' = -\frac{dbz_2}{b} + d\varphi_2 \tag{57b}$$

$$d\xi' = -\frac{h}{b}\,d\kappa_2 - d\omega_2 \tag{57c}$$

These formulas are of fundamental importance for the investigation of the error propagation from the relative orientation after the absolute orientation to the final results of the elevation measurements or to the angles φ and ω of the exterior orientation.

The weight and correlation number of formula (57) can be found as follows, according to the definition:

$$Q_{h_o'h_o'} = Q_{bz_2bz_2} + \frac{(h^2 + b^2)^2}{b^2} Q_{\varphi_2\varphi_2} \; 2 \, \frac{h^2 + b^2}{b} Q_{\varphi_2bz_2} \tag{58a}$$

$$Q_{\eta'\eta'} = \frac{1}{b^2} Q_{bz_2bz_2} + Q_{\varphi_2\varphi_2} - \frac{2}{b} Q_{\varphi_2bz_2} \tag{58b}$$

$$Q_{\xi'\xi'} = \frac{h^2}{b^2} Q_{\kappa_2\kappa_2} + Q_{\omega_2\omega_2} + \frac{2h}{b} Q_{\omega_2\kappa_2} \tag{58c}$$

The weight and correlation numbers of formulas (58) depend upon the method of the relative orientation and the number and location of the model points used (Tables I, II, and V).

Next, the influence upon arbitrary model elevations from formulas (57) can be found after substitution into formula (8). But still in each point, the influence of the relative orientation according to formula (5) is found. Consequently, the influence upon the elevations in arbitrary model points from the relative orientation is found from a sum of formulas (8) and (5).

$$dh_r = dh_o' + x\,d\eta' + y\,d\xi' + \frac{hy}{b} d\kappa_2 - \left(1 - \frac{x}{b}\right) dbz_2 + \frac{h^2 + (x-b)^2}{b} d\varphi_2 - \frac{(x-b)y}{b} d\omega_2 \tag{59}$$

After formulas (57) are substituted and some rearrangements are made, the following is found:

$$dh_r = \frac{x}{b}(x - b)d\varphi_2 - \frac{xy}{b} d\omega_2 \tag{60}$$

The weight number is

$$Q_{h_rh_r} = \frac{x^2}{b^2}(x-b)^2 Q_{\varphi_2\varphi_2} + \frac{x^2y^2}{b^2} Q_{\omega_2\omega_2} - \frac{2x^2y(x-b)}{b^2} Q_{\varphi_2\omega_2} \tag{61}$$

The standard error of unit weight to be used together with this weight number refers to the y-parallax measurements and is denoted s_o. The final standard error of the elevations can now be found from the quadratic summation of formula (53) and $s_{h_r} = s_o \sqrt{Q_{h_rh_r}}$

For the weight and correlation numbers according to Table II we then find:

$$s_h = \sqrt{s_{oh}^2\left(\frac{3}{2} + \frac{3x^2}{2b^2} + \frac{y^2}{2d^2} - \frac{x}{b}\right) + s_o^2\,\frac{x^2h^2}{b^2d^2}\,\frac{(x-b)^2}{b^2} + \frac{3y^2}{4d^2}} \qquad (62)$$

After using the relation formula (54) between s_o and s_{oh} we finally find:

$$s_h = s_o\,\frac{h}{b}\sqrt{\frac{x^2(x-b)^2}{b^2d^2} + \frac{3x^2y^2}{4d^4} + \frac{3x^2}{2b^2} + \frac{y^2}{2d^2} - \frac{x}{b} + \frac{3}{2}} \qquad (63)$$

If the five points, 55, 11, 91, 19, and 99, had been used for the absolute orientation, the following influences of the errors of the relative orientation would have been found upon the elements of the absolute orientation:

$$dh_o' = + \frac{dbz_2}{2} - \left(\frac{h^2}{b} + \frac{9b}{20}\right) d\varphi_2 \qquad (64a)$$

$$d\eta' = - \frac{dbz_2}{b} + d\varphi_2 \qquad (64b)$$

$$d\xi' = - \frac{h}{b}\,d\kappa_2 - \frac{d\omega_2}{2} \qquad (64c)$$

These expressions can be derived after using formulas (9) through (11) and after substituting the individual dh by the corresponding formula (5) applied to the respective points.

After a procedure, identical to the one just shown, we find the following formula for the standard error of the final elevations of the model:

$$s_h = s_o\,\frac{h}{b}\sqrt{\frac{1}{d^2}\left\{\left(-\frac{x^2}{b} + x - \frac{b}{20}\right)^2 + \frac{3(2x-b)^2y^2}{16d^2}\right\} + \frac{6}{5} + \frac{(2x-b)^2}{4b^2} + \frac{y^2}{4d^2}} \quad (65)$$

Formulas (63) and (65) may serve as examples of the standard errors to be expected from photogrammetric elevation measurements under the assumed conditions. A certain generalization of the formulas is possible by using root mean square values of the standard errors over the model surface. Such a root mean square value can be found from the technique for the determination of mean values of functions from the integral calculus. For the area as defined by the model points, 11, 91, 99, and 19, and expressed by 2bd, the root mean square value of the elevation errors can be found from

$$M_h = \sqrt{\frac{1}{2bd}\int_{x=0}^{x=b}\int_{y=-d}^{y=d} s_h^2 \, dxdy} \tag{66}$$

Applying this formula to formulas (63) and (65) we find for the examples of three and five control points:

$$M_{h3} = s_o \sqrt{\frac{7h^2}{60d^2} + \frac{5h^2}{3b^2}} \tag{67}$$

$$M_{h5} = s_o \sqrt{\frac{h^2}{25d^2} + \frac{41h^2}{30b^2}} \tag{68}$$

The correct double integration of formula (68) has been made by Dr. Harkin, GIMRADA.

Formulas (63) and (65) are graphically plotted for $s_o = 1$ and for wide-angle photographs, about 60 percent overlap, in Figs. 7 and 8. For many purposes, it is suitable to simplify formulas (67) and (68) further by substituting the relations $\frac{h}{b}$ and $\frac{h}{d}$ of the photographs. For wide-angle photographs and about 60 percent overlap we find

$$M_{h3} = 2.2s_o \tag{67a}$$

$$M_{h5} = 2.0s_o \tag{68a}$$

In a similar way, as has been briefly shown, the geometrical quality of elevation measurements has been treated for convergent photographs in reference 8. The formulas for the final standard errors of the elevations are identical to those shown previously, and therefore, a comparison between the standard errors of elevation determinations from various kinds of vertical photography (normal angle, wide angle, and superwide angle) and convergent photography can be made. It should be noted that the relation between the flying altitude h and the terms b and d is influenced by the different types of photography, and that consequently, the density of control points on the ground also becomes influenced. Furthermore, the standard error of unit weight of the y-parallaxes must be referred to the same scale as the standard error of the elevations. The geometrical quality of the photogrammetric determinations of elevations on the ground can be determined from information about the relation between h, b, and d, the number and location of the control points, and the standard error of unit weight of the y-parallax measurements.

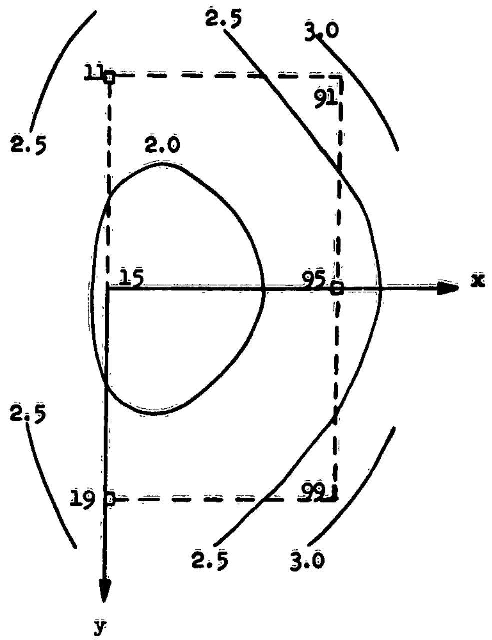

Fig. 7. Standard errors of the final elevations after absolute orientation in the points 11, 19, and 95. Wide-angle photographs, about 60 percent overlap, $s_o = 1$.

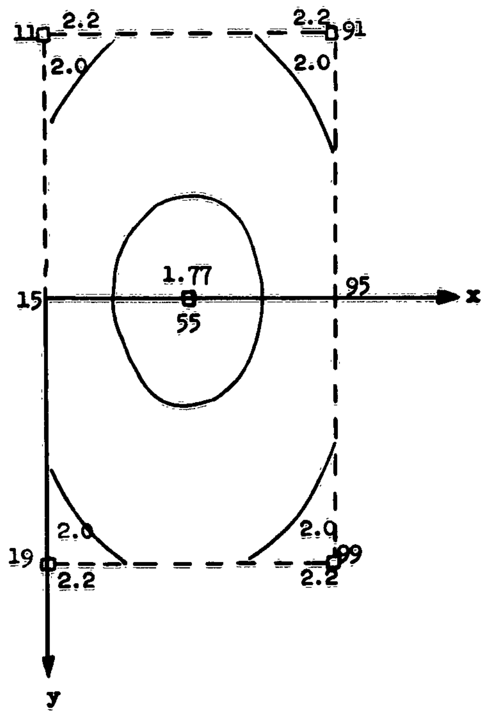

Fig. 8. Standard errors of the final elevations after absolute orientation in the points 55, 11, 91, 19, and 99. Wide-angle photographs, about 60 percent overlap, $s_o = 1$.

It is assumed that the most important regular errors of the photographs and instruments are known and have been corrected for. If the relative orientation is not strictly adjusted according to least squares, the root mean square value of the residual y-parallaxes of the model, as determined from at least 9 but preferably 15 points, can be used as a substitute for the standard error of unit weight. This means a certain approximation.

Some applications of the derived formulas to important problems in topographic mapping, and to the determination of the geometrical quality of the final elements of the exterior orientation are considered here.

b. The Coordinates in Planimetry. In principle, the procedure is applied to two types of control point distribution, viz., two points located in the vicinity of the nadir points of the model (minimum number of points); and five points located in the corners and in the center of the model. The formulas (6), (7), and (12) through (17) are used. The two nadir points are

$$x_{15} = 0 \qquad x_{95} = b$$

$$y_{15} = 0 \qquad y_{95} = 0 \tag{69}$$

Also, here distinction is made between the influence of the coordinate measurements in the model and the model deformations caused by the relative orientation.

If the coordinate errors of the measurements are denoted dx_1, dy_1, dx_2, and dy_2, the differentials (corrections) of the elements of the absolute orientation can be expressed as follows:

$$dx_o = - dx_1 \tag{70a}$$

$$dy_o = - dy_1 \tag{70b}$$

$$db = dx_1 - dx_2 \tag{70c}$$

$$d\alpha = \frac{dy_1 - dy_2}{b} \tag{70d}$$

The weight and correlation numbers are:

$$Q_{x_o x_o} = Q_{y_o y_o} = 1 \tag{71a}$$

$$Q_{bb} = 2 \tag{71b}$$

$$Q_{\alpha\alpha} = \frac{2}{b^2} \tag{71c}$$

$$Q_{x_o b} = -1 \tag{71d}$$

$$Q_{y_o \alpha} = -\frac{1}{b} \tag{71e}$$

The standard errors of corrected x- and y-coordinates according to formulas (12) and (13) are then

$$s_x = s_y = s_{oc}\sqrt{2}\sqrt{\frac{x^2+y^2}{b^2} - \frac{x}{b} + 1} \qquad \text{and} \tag{72}$$

s_{oc} is the standard error of unit weight of the x- or y-coordinate measurements in the model.

Next, the principle of compensation is applied to the model deformation in the control points. The influences of the errors of the relative orientation upon the elements of the absolute orientation are:

$$dx_o = 0 \tag{73a}$$

$$dy_o = \frac{dby_2}{2} + \frac{b}{2}\,d\kappa_2 - \frac{h}{2}\,d\omega_2 \tag{73b}$$

$$db = h\,d\varphi_2 \tag{73c}$$

$$d\alpha = -\frac{d\kappa_2}{2} \tag{73d}$$

The total influence upon the model coordinates of the errors of the relative orientation, including the compensating effect, is as follows:

$$dx = \left(\frac{1}{2} - \frac{x}{b}\right) y\,d\kappa_2 - \frac{x(x-b)}{bh}\,dbz_2 - \frac{x(x-b)^2}{bh}\,d\varphi_2 + \frac{x(x-b)y}{bh}\,d\omega_2 \tag{74}$$

$$dy = -\frac{y^2}{b}\,d\kappa_2 - \left(\frac{x}{b} - \frac{1}{2}\right)\frac{y}{h}\,dbz_2 - \frac{(x-b)^2}{b} + \frac{x-b}{2}\,\frac{y}{h}\,d\varphi_2 + \left(\frac{x}{b} - \frac{1}{2}\right)\frac{y^2}{h}\,d\omega_2 \tag{75}$$

After the general law of error propagation is applied to the preceding formulas, the weight and correlation numbers of Table II are used

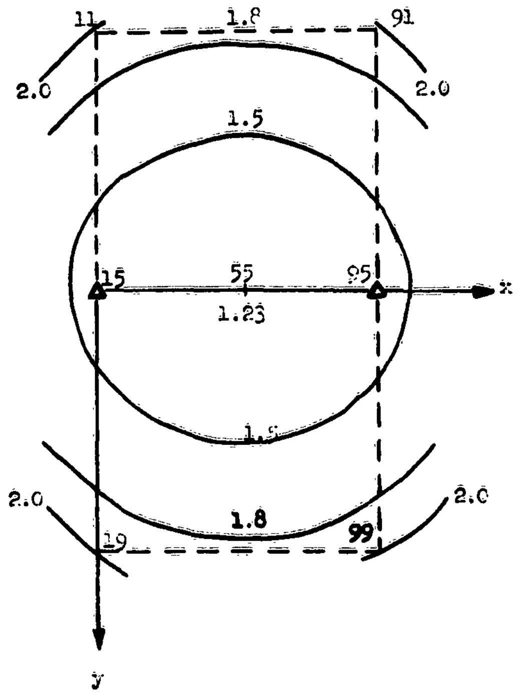

Fig. 9. Standard errors of the final x-coordinates after absolute orientation in the points 15 and 95. Wide-angle photographs, about 60 percent overlap, $s_o = 1$.

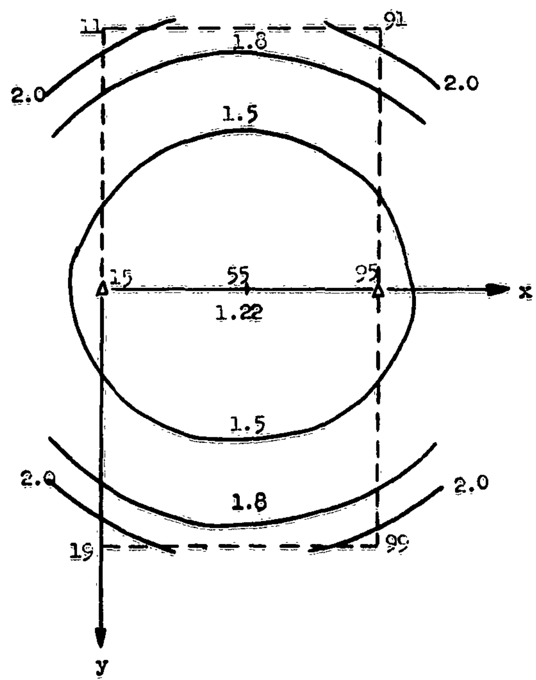

Fig. 10. Standard errors of the final y-coordinates after absolute orientation in the points 15 and 95. Wide-angle photographs, about 60 percent overlap, $s_o = 1$.

and if the standard errors of unit weight of y-parallax measurements and of model coordinate measurements are assumed to be of the same order of magnitude, the following formulas are obtained for the standard errors of the final x- and y-coordinates under the actual conditions:

$$s_x = s_o \sqrt{\frac{(b-2x)^2 y^2}{6b^4} + \frac{x^2(x-b)^2}{2b^2d^2} + \frac{x^2(x-b)^4}{b^4d^2} + \frac{3x^2(x-b)^2y^2}{4b^2d^4} + \frac{x^2(x-b)^3}{b^3d^2} + 2\left(\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{x}{b} + 1\right)} \qquad (76)$$

$$s_y = s_o \sqrt{\frac{2y^4}{3b^4} + \frac{(2x-b)^2y^2}{8b^2d^2} + \left\{\frac{(x-b)^2}{b} + \frac{x-b}{2}\right\}^2 \frac{y^2}{b^2d^2} + \left(\frac{x}{b} - \frac{1}{2}\right)^2 \frac{3y^4}{4d^4} + \left(\frac{x}{b} - \frac{1}{2}\right)\left\{\frac{(x-b)^2}{b} + \frac{x-b}{2}\right\}\frac{y^2}{bd^2} + 2\left(\frac{x^2}{b^2} + \frac{y^2}{b^2} - \frac{x}{b} + 1\right)} \qquad (77)$$

Formulas (76) and (77) are shown graphically in Figs. 9 and 10 for wide-angle photographs, 60 percent overlap, and $s_o = 1$. From the diagrams, the following root mean square values of the standard errors can be found:

$$M_x = 1.6 s_o \qquad (78)$$

$$M_y = 1.7 s_o \qquad (79)$$

For five control points located in the vicinities of the model points, 55, 11, 91, 19, and 99, formulas (14) through (17) are used for the unique determination of the influences of the errors of the relative orientation and of the measurements of the model coordinates upon the elements of the absolute orientation and then upon the final coordinates. The procedure is similar to the one shown previously for two points, only. The final formulas for the standard errors of the x- and y-coordinates are shown here. Some minor errors in the original formulas have been discovered by Dr. Harkin, GIMRADA, and are corrected as follows:

$$s_x = s_o \sqrt{\left(\frac{1}{2} - \frac{x}{b}\right)^2 \frac{2y^2}{3b^2} + \left\{\frac{b}{20} + \frac{x(x-b)}{b}\right\}^2 \frac{1}{2d^2} + \left\{\frac{x(x-b)^2}{b} - \right.}$$

$$\overline{- \left.\frac{xbd^2}{b^2+4d^2} + \frac{16b^2d^2-b^4}{40(b^2+4d^2)}\right\}^2 \frac{1}{b^2d^2} + \left\{\frac{bd^2}{b^2+4d^2} + \frac{x(x-b)}{b}\right\}^2 \frac{3y^2}{4d^4} +}$$

$$\overline{+ \left\{\frac{b}{20} + \frac{x(x-b)}{b}\right\}\left\{\frac{x(x-b)^2}{b} - \frac{xbd^2}{b^2+4d^2} + \frac{16b^2d^2-b^4}{40(b^2+4d^2)}\right\} \frac{1}{bd^2} +}$$

$$\overline{+ \frac{20x^2 - 20bx + 20y^2 + 29b^2 + 96d^2}{20(b^2 + 4d^2)}} \qquad (80)$$

$$s_y = s_o \sqrt{\frac{2(4d^2-5y^2)^2}{75b^4} + \frac{(2x-b)^2y^2}{8b^2d^2} + \left\{\frac{x^2}{b} - \frac{3x}{2} + \frac{b(b^2+2d^2)}{2(b^2+4d^2)}\right\}^2 \frac{y^2}{b^2d^2} +}$$

$$\overline{+ \left\{- \frac{xy^2}{b} + \frac{y^2}{2} + \frac{(2x-b)bd^2}{2(b^2+4d^2)}\right\}^2 \frac{3}{4d^4} + \frac{(2x-b)y^2}{2b^2d^2} \left\{\frac{x^2}{b} - \frac{3x}{2} + \right.}$$

$$\overline{+ \left.\frac{b(b^2+2d^2)}{2(b^2+4d^2)}\right\} + \frac{20x^2 - 20bx + 20y^2 + 29b^2 + 96d^2}{20(b^2 + 4d^2)}} \qquad (81)$$

Formulas (80) and (81) are shown graphically in Figs. 11 and 12 for wide-angle photographs, 60 percent overlap, and $s_o = 1$. The root mean square values of the standard error of the x- and y-coordinates are

$$M_{s_x} = M_{s_y} = 1.2s_o \qquad (82)$$

4. Summary of the Investigations of the Relative and Absolute Orientations and Some Results of Practical Tests. The derived formula systems show the accuracy of the final results of the photogrammetric determination of numerical values of elevations and coordinates in planimetry from vertical photographs under the assumed conditions. Again, it should be emphasized that the operations numbers 1 and 2 of the photogrammetric procedure, the photography, and the reconstruction of the bundles of rays, must be tested for regular

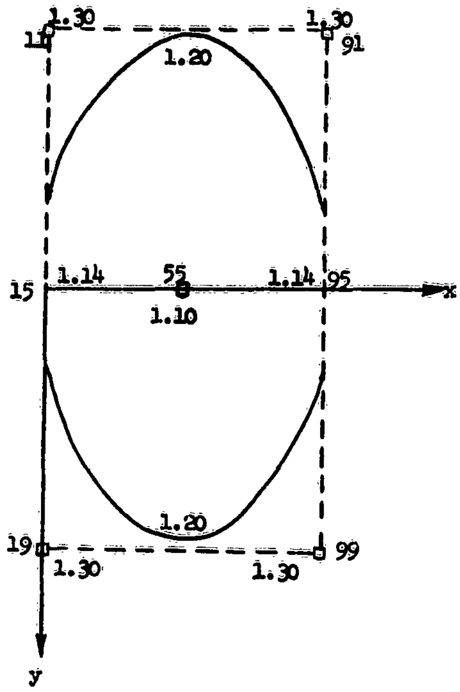

Fig. 11. Standard errors of the x-coordinates after absolute orientation in the points 55, 11, 91, 19, and 99. Wide-angle photographs, about 60 percent overlap, $s_0 = 1$.

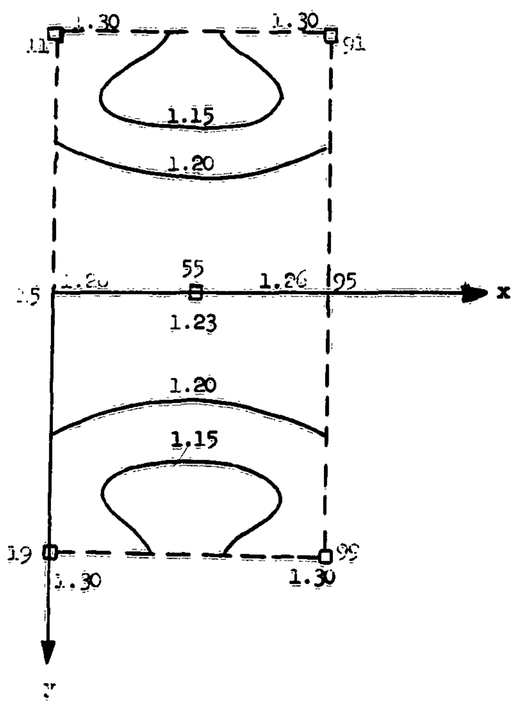

Fig. 12. Standard errors of the y-coordinates after absolute orientation in the points 55, 11, 91, 19, and 99. Wide-angle photographs, 60 percent overlap, $s_0 = 1$.

errors under operational conditions and that the regular errors are corrected for up to the tolerance limits according to the irregular errors. It is further assumed that the latter errors are normally distributed at least at the 5 percent confidence level. Moreover, if the relative and the absolute orientations are treated as previously stated, i. e., in particular, the inevitable discrepancies of these operations are adjusted according to least squares, the accuracy of the final results can be expressed in terms of the standard error of the y-parallax observations. With some approximation the root mean square values of the residual y-parallaxes after the absolute orientation can be used for the accuracy determination.

As is shown here, the derived formula systems can be used for the treatment of important practical problems in connection with the planning of aerial photography and determinations of the necessary density of ground control under certain requirements concerning the final accuracy in elevation and planimetry. Furthermore, the accuracy of the elements of the exterior orientation can be determined theoretically and uniquely. This determination is of great importance because photogrammetric methods frequently are used for tests of other methods for the determination of the space coordinates, e. g., of airplanes. In such tests, the accuracy of the checking method must be well known. Before these applications are discussed, some results of practical tests of the formulas derived previously are shown.

The most important tests were performed in connection with the controlled experiments within the International Society of Photogrammetry during the period 1956 through 1960. From series of y-parallax measurements in connection with the practical photogrammetric plotting and coordinate determination, the accuracy to be expected in the final results was predicted by the Subcommission IV:4. The tests were applied to large-scale plotting after aerial photography of rugged terrain from a comparatively low altitude (Subcommission IV:1 I.S.P.), as well as to small-scale mapping after photography from high altitude (8,000 m, Subcommission IV:3). The results of the tests are presented in the report from Subcommission IV:4 at the London Congress in 1960 (reference 5). The agreement between the predictions according to the theoretical formulas and the practical results were astonishingly good. The discrepancies were found to be well within the tolerance limits at the 5 percent level, and the theory behind the formulas, therefore, became verified. This means that comparatively simple y-parallax measurements can be used for a good check on the accuracy to be expected from photogrammetric plotting under assumed conditions.

Further checks have been made in different connections. In particular, one made within GIMRADA follows.

For certain tests, elevation measurements had been made in a great number of check points in 24 wide-angle models. The residual y-parallaxes had also been measured in 9 points of each model. The y-parallax measurements were made available and a root mean square value of 9 microns was determined. In each model, four control points had been used for the absolute orientation. A theoretical investigation was made concerning the error propagation under the actual conditions, and the root mean square value of the errors to be expected in the final elevations was computed as a function of the root mean square value of the residual y-parallaxes. The following formula was found:

$$M_h = 2.0 s_{m9} S$$

where M_h is the root mean square value of the elevation errors to be expected;

s_{m9} is the root mean square value of the y-parallaxes (the standard error of unit weight of the y-parallaxes if a strict adjustment had been made) on the scale of the photographs;

S is the scale factor of the photographs, i. e., $\frac{h}{c}$ where h is the flying altitude and c is the camera constant.

Here, the scale factor S was 18,000 and the expected value of M_h became, consequently:

$$M_h = 2.0 \times 9 \times 18{,}000 \text{ microns} = 0.324 \text{ m}$$

The confidence limits at the 5 percent level are 0.194 through 0.940 m. The root mean square value of the elevation discrepancies in a great number of check points was found to be 0.344 m. The agreement between theory and practice is consequently very good, and another practical verification of the theoretical laws of error propagation has been obtained.

Even if there are a great number of similar verifications, all showing a good agreement between theory and practice, still more practical tests are desirable. It is always advisable to measure the residual y-parallaxes independently in connection with practical photogrammetrical work and, in particular, in connection with ordinary photogrammetric tests.

5. <u>Summary of Formulas for the Standard Errors in Planimetry and Elevation of Various Methods of Photogrammetric Triangulation</u>. Similar principles as have been applied previously for the derivation of formulas for the geometrical quality in planimetry and elevation from single models can be used for investigations of the

geometrical quality of the results of various methods for photogrammetric triangulation. Necessary prerequisites have been clearly stated and are summarized here. The fundamental operations must be carefully investigated and tested for the distinction between regular and irregular errors. The irregular ones are to be estimated as standard errors of unit weight according to the method of least squares. The residuals after the adjustments, from which the standard error is computed, must be normally distributed at a reasonable level, usually 5 percent. Discrepancies in available conditions, primarily those of the relative orientation shall, in principle, be adjusted according to the method of least squares. It is further assumed that the elevation differences of the ground are moderate and that the aerial photography is made under normal, uniform conditions.

Here, we distinguish between three methods of photogrammetric triangulation which are of practical importance (reference 6).

a. Ordinary Aerial Triangulation. No auxiliary information concerning the elements of the exterior orientation is assumed to be available. For wide-angle photographs and 60-percent overlap the standard errors in planimetry and elevation can be expressed by the following formulas.

(1) Cantilever Extension.

$$s_x = s_o \frac{h}{c} \sqrt{\frac{1}{6}(4n^3 + 63n^2 - 10n + 12)}$$

$$s_y = s_o \frac{h}{c} \sqrt{\frac{1}{18}(4n^3 + 39n^2 - 31n + 39)}$$

$$s_z = s_o \frac{h}{c} \sqrt{n^3 + 6n^2 - 10n + 33}$$

Only the first model is assumed to be absolutely oriented with control points.

n is the number of models after the first one;

h is the flying altitude;

c is the camera constant; and

s_o is the standard error of unit weight of the y-parallaxes and image coordinates.

In Table VI, some examples of the standard errors are shown under well-defined assumptions. The standard errors in planimetry are presented as radial standard errors but the possible correlation is not taken into account.

Table VI. Standard Errors in Planimetry and Elevation after Cantilever Extension; Wide-Angle Photographs, 60-Percent Overlap

Number of Models n	Standard Errors on the Scale of the Photographs for $s_o = 1$		Standard Errors for s_o = 0.01 mm and h:c = 20,000 Meters		Triangulated Distances	
	$s_{p(xy)}$	s_z	$s_{p(xy)}$	s_z	Kilometers	Miles
3	11.49	9.17	2.2	1.8	3.7	2.3
4	15.82	12.37	3.2	2.5	5.5	3.4
5	20.37	16.06	4.1	3.2	7.4	4.5
6	25.14	20.12	5.0	4.0	9.2	5.7
7	30.10	25.48	6.0	5.1	11.0	6.8
8	35.25	29.13	7.0	5.9	12.9	8.0
9	40.59	34.03	8.1	6.8	14.7	9.1
10	46.11	39.15	9.2	7.8	16.6	10.3
15	76.18	67.88	15.2	13.6	25.8	16.0
20	110.06	101.16	22.0	20.2	35.0	21.8
25	147.39	138.41	29.5	27.7	44.1	27.4
50	377.64	373.54	75.5	74.7	90.2	56.0

In long triangulation strips, longer than at least 10 models, the arc sine law becomes actual. This law states that even if there are only irregular, normally distributed errors in the basic observations of the aerial triangulation the errors in planimetry and elevation along the strip may show apparent systematic or regular variations. These are caused by the correlation in the summation and are not regular errors caused by regular sources of errors. Sometimes, therefore, the irregular errors, expressed as standard errors according to Table VI, may be considerably influenced by the apparent regular errors according to the arc sine law in long strips. Therefore, even if the standard errors theoretically would amount to the figures shown in this table, the discrepancies found in practice from a long triangulation strip can deviate more from the theoretical values than the confidence limits allow. Under all circumstances, it is advisable to keep triangulation strips between control points below 10 models because of the just mentioned arc sine law. It should also be remembered that the previous formulas were obtained

under well-defined conditions and that some approximations have been used.

(2) Bridging. The bridging is made between two groups of control points at the ends of the strip.

According to reference 6, the standard errors in planimetry and elevation can be expressed by the following formulas, which have been derived under well-defined conditions and with certain approximations.

$$s_x = s_o \frac{h}{c} \sqrt{\frac{p(n-p)}{3n} \left\{2p(n-p) + 1\right\}}$$

$$s_y = s_o \frac{h}{3c} \sqrt{\frac{2p(n-p)}{n} \left\{p(n-p) + 2\right\}}$$

$$s_z = s_o \frac{5h}{3c} \sqrt{\frac{p(n-p)}{6n} \left\{2p(n-p) + 13\right\}}$$

In these formulas, the strip is assumed to consist of the photographs

$$-1,\ 0,\ +1,\ 2 \ldots\ldots\ldots\ n-1,\ n,\ n+1$$

The control points are located in the models -1, 0, and n, n+1, and p is the notation for an arbitrary model, the standard errors of which are to be determined. The other notations are identical with the notations used in the cantilever extension method.

In Table VII, the previous formulas are numerically calculated for some values of the coefficients. The standard error in planimetry is expressed as a radial standard error but no attention is given to the possible correlation between x and y.

Also, here the arc sine law must be remembered. Triangulations over more than 10 models may be influenced by the correlation in connection with the summation of the errors from the fundamental operations. The confidence intervals can be determined by statistical methods for certain levels and with respect to the number of observations in the determination of the basic standard errors of unit weight.

b. Independent Model Triangulation. In this form of triangulation, the individual models are levelled with the aid of available control. The triangulation is performed through a series of planimetry coordinate transformations from model to model. Only

Table VII. Standard Errors in Planimetry and Elevation after Bridging, Referred to the Middle of the Strip

Number of Models		Standard Errors on the Scale of the Photographs for $s_o = 1$		Standard Errors for s_o = 0.01 mm and h:c = 20,000 Meters		Triangulated Distances	
n	n+3	$s_{p(xy)}$	s_z	$s_{p(xy)}$	s_z	Kilometers	Miles
4	7	3.87	3.12	0.8	0.6	5.5	3.4
5	8	5.12	3.84	1.0	0.8	7.4	4.5
6	9	6.52	4.64	1.3	0.9	9.2	5.7
7	10	8.05	5.51	1.6	1.1	11.0	6.8
8	11	9.70	6.46	1.9	1.3	12.9	8.0
9	12	11.46	7.47	2.2	1.5	14.7	9.1
10	13	13.32	8.54	2.7	1.7	16.6	10.3
15	18	24.06	14.76	4.8	3.0	25.8	16.0
20	23	36.81	22.21	7.4	4.4	35.0	21.8
25	28	51.30	30.70	10.2	6.1	44.1	27.4
50	53	144.54	85.51	28.9	17.1	90.2	56.0

coordinates in planimetry are to be determined through this form of triangulation. Also, here, distinction is made between cantilever extension and bridging. The following formulas have been derived (reference 6):

(1) Cantilever Extension.

$$s_x = s_o \frac{h}{c} \sqrt{\frac{1}{3} (20\ n^3 + 54\ n^2 + 61\ n + 3)}$$

$$s_y = s_o \frac{h}{c} \sqrt{\frac{1}{18} (10\ n^3 + 6\ n^2 + 50\ n + 18)}$$

(2) Bridging.

$$s_x = s_o \frac{h}{c} \sqrt{\frac{p(n-p)}{3n} \left\{20p(n-p) + 61\right\}}$$

$$s_y = s_o \frac{h}{c} \sqrt{\frac{p(n-p)}{9n} \left\{5p(n-p) + 25\right\}}$$

The notations are identical with those used in the discussion of ordinary triangulation just given.

c. Numerical Radial Triangulation. For a numerical radial triangulation according to reference 6, the error propagation has been derived for cantilever extension and for bridging. Here, the basic standard error of unit weight refers to the image coordinates and furthermore, the radial distortion in the photographs is of limited importance for the accuracy of the triangulation results. It must be assumed, on the other hand, that the inclinations of the photographs are sufficiently small so that the influence upon the image coordinates can be neglected. This condition can be checked in connection with the measurements of the image coordinates through determination of the y-parallaxes between adjacent photographs. From the y-parallaxes the inclination differences between the photographs in a strip can be determined and taken into account if necessary. The nadir point or the isocenter can be determined or, still more suitable, numerical corrections to the measured image coordinates can be computed from the determined inclinations and inclination differences. The simplest method is to use a mirror stereoscope and a parallax bar for the basic parallax measurements. Through such simple methods the requirement on sufficient verticality can usually be fulfilled for the following formula systems:

(1) Cantilever Extension.

$$s_x = s_o \frac{h}{c} \sqrt{\frac{n}{6}(14n^2 + 21\,n + 25)}$$

$$s_y = s_o \frac{h}{c} \sqrt{\frac{n}{6}(10\,n^2 + 15\,n + 107)}$$

(2) Bridging.

$$s_x = s_o \frac{h}{c} \sqrt{\frac{p(n-p)}{6n}\left\{14p(n-p) + 25\right\}}$$

$$s_y = s_o \frac{h}{c} \sqrt{\frac{p(n-p)}{6n}\left\{10p(n-p) + 107\right\}}$$

Experiments have proved that the standard error of unit weight, which in principle should refer to image coordinates, can be referred to the y-parallaxes and can be determined in connection with the measurements of such parallaxes for the determination of

the inclination differences. Consequently, the theory of errors of the relative orientation can be applied here also, although in a limited sense. Because a combination between stereoscopic measuring methods and radial methods is assumed, the actual triangulation method has been denoted stereoradial triangulation.

6. Determination of the Accuracy of the Elements of the Exterior Orientation after Double-Point Resection in Space.

a. The Influence of the Relative and Absolute Orientation Procedures. From the basic formulas derived previously, the accuracy of the elements of the exterior orientation can be determined. The accuracy will, of course, depend upon the number of control points and their locations. The derivation is referred to the minimum number of control points and locations as assumed previously. Derivations of the final accuracy for other combinations and locations of control points can be made according to the same principles.

The final values of the angles φ and ω are denoted $\bar{\varphi}_1$ $\bar{\omega}_1$ and $\bar{\varphi}_2$ $\bar{\omega}_2$, respectively. The final values of the three space coordinates of the exposure stations are denoted X_1, Y_1, Z_1 and X_2, Y_2, Z_2, respectively. For the determination of the accuracy, the differentials of these data only are of interest. The values of the preliminary results of the resection in space can be determined with the aid of the scales of the instrument or from numerical calculations.

According to Figs. 1 and 2 and because the relative orientation is assumed to be made according to the dependent procedure, the following relations are immediately found:

$$d\bar{\varphi}_1 = - d\eta \tag{83}$$

$$d\bar{\varphi}_2 = d\varphi_2 - d\eta \tag{84}$$

$$d\bar{\omega}_1 = d\xi \tag{85}$$

$$d\bar{\omega}_2 = d\omega_2 + d\xi \tag{86}$$

Next, the expressions for $d\eta$ and $d\xi$ are substituted from formulas (51) and (57).

$$d\bar{\varphi}_1 = \frac{2dh_{95} - dh_{11} - dh_{19}}{2b} + \frac{dbz_2}{b} - d\varphi_2 \tag{87}$$

$$d\bar{\varphi}_2 = \frac{2dh_{95} - dh_{11} - dh_{19}}{2b} + \frac{dbz_2}{b} \tag{88}$$

$$d\bar{\omega}_1 = \frac{dh_{19} - dh_{11}}{2d} - \frac{h}{b}\, d\varkappa_2 - d\omega_2 \tag{89}$$

$$d\bar{\omega}_2 = \frac{dh_{19} - dh_{11}}{2d} - \frac{h}{b}\, d\varkappa_2 \tag{90}$$

The standard errors of the angles are now found from the laws of error propagation. The standard error of $\bar{\varphi}_1$ is derived, e. g. We find:

$$s^2_{\bar{\varphi}_1} = s^2_{oh}\, \frac{3}{2b^2} + s^2_o \left(\frac{Q_{bz_2bz_2}}{b^2} + Q_{\varphi_2\varphi_2} - \frac{2}{b}\, Q_{bz_2\varphi_2} \right)$$

According to Table II and formula (54), we find:

$$s_{\bar{\varphi}_1} = s_o\, \frac{h}{b} \sqrt{\frac{1}{2d^2} + \frac{3}{2b^2}} = s_{\bar{\varphi}_2} \tag{91}$$

Similarly, the following is found:

$$s_{\bar{\omega}_1} = s_o\, \frac{h}{b} \sqrt{\frac{2}{3b^2} + \frac{1}{2d^2} + \frac{3b^2}{4d^4}} \tag{92}$$

$$s_{\bar{\omega}_2} = s_o\, \frac{h}{b} \sqrt{\frac{1}{3b^2} + \frac{1}{2d^2}} \tag{93}$$

For five elevation control points the following formulas are found:

$$s_{\bar{\varphi}_1} = s_{\bar{\varphi}_2} = s_o\, \frac{h}{b} \sqrt{\frac{1}{2d^2} + \frac{1}{b^2}} \tag{94}$$

$$s_{\bar{\omega}_1} = s_{\bar{\omega}_2} = s_o\, \frac{h}{b} \sqrt{\frac{2}{3b^2} + \frac{3b^2}{16d^4} + \frac{1}{4d^2}} \tag{95}$$

Concerning the three space coordinates of each exposure station, we have the differential relations (Figs. 1 and 2 and reference 9):

$$dX_1 = dx_o - hd\eta \tag{96}$$

$$dX_2 = dx_o - hd\eta + db \tag{97}$$

$$dY_1 = dy_o - hd\xi \tag{98}$$

$$dY_2 = dy_o - hd\xi + bd\alpha \tag{99}$$

$$dZ_1 = dh_o \tag{100}$$

$$dZ_2 = dh_o + bd\eta - dbz_2 \tag{101}$$

For six orientation points for the relative orientation and minimum number of control points (two in planimetry and three in elevation) in the locations previously assumed, we find from formulas (51), (57), (70), and (73):

$$dX_1 = -dx_1 - \frac{h}{2b}(2dh_{95} - dh_{11} - dh_{19}) - hd\varphi_2 + \frac{h}{b}dbz_2 \tag{102}$$

$$dX_2 = -dx_2 - \frac{h}{2b}(2dh_{95} - dh_{11} - dh_{19}) + \frac{h}{b}dbz_2 \tag{103}$$

$$dY_1 = -dy_1 - \frac{h}{2d}(dh_{19} - dh_{11}) + \frac{h}{2}d\omega_2 + \frac{2h^2+b^2}{2b}d\kappa_2 + \frac{dby_2}{2} \tag{104}$$

$$dY_2 = -dy_2 - \frac{h}{2d}(dh_{19} - dh_{11}) + \frac{h}{2}d\omega_2 + \frac{h^2}{b}d\kappa_2 + \frac{dby_2}{2} \tag{105}$$

$$dZ_1 = \frac{1}{2}(dh_{11} + dh_{19}) + dbz_2 - \frac{h^2 + b^2}{b}d\varphi_2 \tag{106}$$

$$dZ_2 = dh_{95} - dbz_2 - \frac{h^2}{b}d\varphi_2 \tag{107}$$

The standard errors of the space coordinates are then found with the aid of the general law of error propagation. The weight and correlation numbers of the elements of the relative orientation are obtained from Table II. For the coordinate X_1 we find:

$$s^2_{X_1} = s^2_{oc} + s^2_{oh}\frac{3h^2}{2b^2} + s^2_o\frac{h^4}{2b^2d^2} \tag{108}$$

where

s_{oc} is the standard error of unit weight of the model coordinate measurements;

s_{oh} is the standard error of unit weight of the model elevation measurements; and

s_o is the standard error of unit weight of the y-parallax measurements.

When the following assumptions are introduced

$$s_{oc} = s_o$$

$$s_{oh} = \frac{h}{b} s_o$$

the following expressions for the standard errors of the coordinates of the exterior orientation, resulting from the relative and absolute orientation, are found:

$$s_{X_1} = s_{X_2} = s_o \sqrt{1 + \frac{3h^4}{2b^4} + \frac{h^4}{2b^2d^2}} \tag{109}$$

$$s_{Y_1} = s_o \sqrt{\frac{7}{6} + \frac{2h^4}{3b^4} + \frac{3h^4}{4d^4} + \frac{h^4}{2b^2d^2} + \frac{h^2}{3b^2} + \frac{h^2}{2d^2}} \tag{110}$$

$$s_{Y_2} = s_o \sqrt{\frac{7}{6} + \frac{2h^4}{3b^4} + \frac{h^4}{2b^2d^2} + \frac{h^2}{3b^2}} \tag{111}$$

$$s_{Z_1} = s_o \sqrt{\frac{h^6}{b^4d^2} + \frac{h^4}{b^2d^2} + \frac{h^2}{2b^2} + \frac{h^2}{2d^2}} \tag{112}$$

$$s_{Z_2} = s_o \sqrt{\frac{h^6}{b^4d^2} + \frac{h^4}{b^2d^2} + \frac{h^2}{b^2} + \frac{h^2}{2d^2}} \tag{113}$$

The differences between the standard errors of the Y- and Z-coordinates of the exposure stations 1 and 2 are caused by the assumed number and locations of the control points. Similar derivations can be made for arbitrary conditions.

b. <u>The Influence of the Elements of the Interior Orientation of Cameras and Projectors</u>. In reference 1, the theory of errors of the interior orientation has been discussed and, in particular, the standard errors of the elements of the interior orientation have been derived from the calibration procedure. A well-defined procedure for the calibration is necessary for the determination of the actual standard errors as functions of the standard errors of unit

weight of the image coordinates. From the derivations made in reference 1, the following standard errors were found for the principal distance and the image coordinates of the principal point:

$$s_c = s_o' \frac{c}{2r'} \tag{114}$$

$$s_{x_p} = s_{y_p} = s_o' \frac{2\sqrt{3}}{3} \tag{115}$$

The standard errors of the angles φ' and ω' of the image from the calibration procedure were found to be

$$s_{\varphi'} = s_{\omega'} = s_o' \frac{c}{r'^2}\sqrt{\frac{5}{6}} \tag{116}$$

In these formulas s_o' is the standard error of unit weight of the image coordinates as determined according to the method of least squares from an adjustment of discrepancies in points on a circle with the radius r' around the principal point and one point in the vicinity of the principal point. When the formulas are applied to aerial photographs taken under operational conditions, the standard error of unit weight should refer to tests under such conditions.

It should be noted that the final camera constant (calibrated focal length) is determined after a correction in order to make the radial distortion zero along a circle around the principal point with the radius r_o' (the zero distortion radius). The correction is computed from the expression

$$dc = \frac{c}{r_o'}\, dr_o' \tag{117}$$

The standard error of the radial distortion amount is about $\frac{s_o'}{2}$ under the conditions assumed in reference 1 concerning the number of points, and the like. The standard error of formula (117) is:

$$s_{dc} = \frac{s_o'\, c}{2\, r_o'}$$

The standard error of the correction dc, therefore, is dependent upon the radius of the circle to be chosen as zero-circle concerning the radial distortion. This depends upon the shape of the radial distortion curve and, consequently, also upon the optical system of the camera (sometimes also upon the optical system of the plotting instrument). Therefore, because the standard error of the correction

obviously is comparatively small in comparison with other standard errors of the interior orientation, it is neglected here.

The standard error of the coordinates of the principal point in the system of the fiducial marks is also affected with errors resulting from the measurements in the fiducial marks. We assume here that the standard errors of these measurements are of the same magnitude as s_o'. Furthermore, the standard errors of the elements of the interior orientation are transferred to the elements of the exterior orientation with the scale factor of the photographs $\frac{h}{c}$.

The standard errors of the elements of the exterior orientation caused by the elements of the interior orientation are consequently, with minor approximations:

$$s_{X'} = s_{Y'} = s_o' \frac{h}{c} \sqrt{\frac{7}{3}} \approx 1.5 s_o' \frac{h}{c} \tag{118}$$

$$s_{Z'} = s_o' \frac{h}{2r'} \tag{119}$$

Further, we have

$$s_{\varphi'} = s_{\omega'} = s_o' \frac{c}{r'^2} \sqrt{\frac{5}{6}} \tag{120}$$

Formulas (118) through (120) refer to the photographs only and are therefore to be used in analytical determinations of the elements of the exterior orientation. If conventional plotting instruments are used for the purpose, the influence of the errors of the projectors of the instruments has to be taken into account. When the same calibration procedure is assumed to be applied to the projectors as to the cameras (the grid method), the influence of the errors of the calibration procedure upon the elements of the exterior orientation become of the same type as has been shown previously concerning the camera (photographs) in formulas (118) through (120). According to experience from practical tests (references 1, 2, and 5) the standard errors of unit weight seem to be of the same order of magnitude, for the photographs as well as for the projectors, but great variations may occur in photographs and instruments used in practice. It is most important that practical tests of the basic accuracy be made of all cameras (photographs) and plotting instruments to be used for the determination of the accuracy of the elements of the exterior orientation.

In order to derive the final formulas for the accuracy of the elements of the exterior orientation, it is necessary to

distinguish between measurements in stereocomparators and analytical calculations, on one hand, and measurements in stereoscopic plotters, on the other. In the latter, the errors of the projectors must also be included in the formulas. The standard errors expressed by formulas (118) through (120) will, under the assumptions previously made, be multiplied by $\sqrt{2}$ for the second method. The final standard errors of the elements of the exterior orientation after measurements in a stereoscopic plotter can now be expressed as follows:

$$s_{X_1} = s_{X_2} = \frac{h}{c} \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{b^2}{h^2} + \frac{3h^2}{2b^2} + \frac{h^2}{2d^2}\right) + \frac{14}{3} s_o'^2} \tag{121}$$

$$s_{Y_1} = \frac{h}{c} \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{1}{3} + \frac{7b^2}{6h^2} + \frac{b^2}{2d^2} + \frac{h^2}{2d^2} + \frac{2h^2}{3h^2} + \frac{3b^2h^2}{4d^4}\right) + \frac{14}{3} s_o'^2} \tag{122}$$

$$s_{Y_2} = \frac{h}{c} \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{1}{3} + \frac{7b^2}{6h^2} + \frac{2h^2}{3b^2} + \frac{h^2}{2d^2}\right) + \frac{14}{3} s_o'^2} \tag{123}$$

$$s_{Z_1} = \frac{h}{c} \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{1}{2} + \frac{b^2}{2d^2} + \frac{h^2}{d^2} + \frac{h^4}{b^2d^2}\right) + \frac{c^2}{2r'^2} s_o'^2} \tag{124}$$

$$s_{Z_2} = \frac{h}{c} \sqrt{s_o^2 \frac{h^2}{b^2}\left(1 + \frac{b^2}{2d^2} + \frac{h^2}{d^2} + \frac{h^4}{b^2d^2}\right) + \frac{c^2}{2r'^2} s_o'^2} \tag{125}$$

$$s_{\bar{\varphi}_1} = s_{\bar{\varphi}_2} = \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{1}{2d^2} + \frac{3}{2b^2}\right) + \frac{5c^2}{3r'^4} s_o'^2} \tag{126}$$

$$s_{\bar{\omega}_1} = \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{2}{3b^2} + \frac{1}{2d^2} + \frac{3b^2}{4d^4}\right) + \frac{5c^2}{3r'^4} s_o'^2} \tag{127}$$

$$s_{\bar{\omega}'_2} = \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{2}{3b^2} + \frac{1}{2d^2}\right) + \frac{5c^2}{3r'^4} s_o'^2} \tag{128}$$

In a similar way, the standard error of the direction (azimuth) of the base can be determined. We find

$$s_y = \sqrt{s_o^2 \frac{h^2}{b^2}\left(\frac{5}{2h^2} + \frac{1}{d^2} + \frac{3h^2}{4d^4}\right) + \frac{28}{3b^2} s_o'^2} \quad (129)$$

According to practical experience from tests in connection with calibrations, the standard errors of unit weight of the image coordinates s_o' and of the y-parallaxes s_o can be treated as approximately equal, and consequently, formulas (121) through (129) can be simplified. Under this assumption, the numerical values of the standard errors as shown in Table VIII have been computed. Distinction has been made between three types of photographs, viz., normal angle, wide angle, and superwide angle. The camera constants are 210 mm, 153 mm, and 88 mm, respectively, and the image formats 180 by 180 mm, 230 by 230 mm, and 230 by 230 mm. The following values of r' have been assumed: 120 mm, 153 mm, and 146 mm, respectively. The standard errors of unit weight of the y-parallaxes and of the image coordinates have been obtained from different sources. For the normal and wide-angle photographs primarily, the results from the International Controlled Experiments from 1956 through 1960, as reported in reference 5, have been used in addition to the results of tests from 1952 through 1956, as reported to the Sub-commission IV at the Stockholm Congress for photogrammetry in 1956. The standard errors of unit weight are 0.004 and 0.006 mm, respectively. For superwide-angle photographs there are only a few practical tests of the y-parallaxes available. The standard error of unit weight to be used in Table VIII is 0.010 mm and has been obtained during some test measurements within GIMRADA. It should be emphasized that the standard error of unit weight can be used only under the condition that the orientation is adjusted according to the method of least squares. If empirical adjustment according to usual methods is used, the standard error of unit weight should be substituted by the root mean square value of the residual y-parallaxes as determined from at least nine but preferably 15 points. Practical experience has indicated that the root mean square value usually is about twice the standard error of unit weight. The figures of this table, therefore, should be regarded to show the highest quality to be expected in the determination of the elements of the exterior orientation from double-point resection in space under assumed conditions. It is always advisable to measure and record the residual y-parallaxes after the absolute orientation has been finished. The standard errors of the angles are given in units of centesimal minutes (^{c}). If s_o is given in units of millimeters, the standard errors of the angles are obtained in units of centesimal seconds (^{cc}) of which there are 100 to a minute (^{c}).

For a certain flying altitude the wide-angle photographs give the best results, except for the flying altitudes, where the superwide-angle photographs are better. It should be emphasized

again that the standard error of unit weight of superwide-angle photographs should be better determined.

Table VIII. Standard Errors of the Elements of the Exterior Orientation after Double-Point Resection in Space in a First-Order Photogrammetric Instrument

Photographs	$s_{X_1} = s_{X_2}$	s_{Y_1}	s_{Y_2}	s_{Z_1}	s_{Z_2}	$s_{\bar{\varphi}_1} = s_{\bar{\varphi}_2}$	$s_{\bar{\omega}_1}$	$s_{\bar{\omega}_2}$
Normal angle $c = 210$ mm $s_o = 0.004$ mm	$12.9 s_o \frac{h}{c}$ $\frac{h}{4100}$	$13.0 s_o \frac{h}{c}$ $\frac{h}{4000}$	$10.2 s_o \frac{h}{c}$ $\frac{h}{5100}$	$28.6 s_o \frac{h}{c}$ $\frac{h}{1800}$	$28.7 s_o \frac{h}{c}$ $\frac{h}{1800}$	$40110 s_o$ $1^c.6$	$39470 s_o$ $1^c.6$	$31831 s_o$ $1^c.3$
Wide angle $c = 153$ mm $s_o = 0.006$ mm	$4.6 s_o \frac{h}{c}$ $\frac{h}{5500}$	$4.9 s_o \frac{h}{c}$ $\frac{h}{5200}$	$4.1 s_o \frac{h}{c}$ $\frac{h}{6200}$	$5.7 s_o \frac{h}{c}$ $\frac{h}{4500}$	$5.8 s_o \frac{h}{c}$ $\frac{h}{4400}$	$17190 s_o$ $1^c.0$	$16870 s_o$ $1^c.0$	$13560 s_o$ $0^c.8$
Superwide angle $c = 88$ mm $s_o = 0.010$ mm	$2.8 s_o \frac{h}{c}$ $\frac{h}{3200}$	$2.9 s_o \frac{h}{c}$ $\frac{h}{3000}$	$2.8 s_o \frac{h}{c}$ $\frac{h}{3200}$	$1.8 s_o \frac{h}{c}$ $\frac{h}{4900}$	$1.9 s_o \frac{h}{c}$ $\frac{h}{4600}$	$10680 s_o$ $1^c.1$	$10580 s_o$ $1^c.1$	$8500 s_o$ $0^c.8$

7. Determination of the Accuracy of the Elements of the Exterior Orientation after Single-Point Resection in Space. For completeness sake, the formulas for the accuracy of the elements of the exterior orientation after single-point resection in space are summarized here. The principles of the derivation have been used in references 9 and 10. The basic differential formulas for the relation between small errors of the elements of the exterior orientation and the corresponding errors of the projected coordinates are well known and have been shown in formulas (1) and (2). These formulas are applied to three points with the following coordinates:

$$\begin{array}{lll} x_1 = a & x_2 = -a & x_3 = -a \\ y_1 = 0 & y_2 = -a & y_3 = a \end{array} \qquad (130)$$

Through a preliminary resection in space the elements of the exterior orientation are assumed to have been rather well determined and only small discrepancies remain. The six equations, which

can be formed with the aid of the six coordinates, can be solved in a general manner, and the corrections of the elements of the exterior orientation can be expressed as linear functions of the discrepancies. Then, the weight and correlation numbers can be determined according to their definitions. We find:

$$Q_{x_o x_o} = 1 + \frac{7c^4}{8a^4} + \frac{3c^2}{2a^2} \tag{131}$$

$$Q_{y_o y_o} = \frac{5}{9} + \frac{7c^4}{18a^4} + \frac{5c^2}{9a^2} \tag{132}$$

$$Q_{z_o z_o} = \frac{3c^2}{8a^2} \tag{133}$$

$$Q_{\varkappa\varkappa} = \frac{2c^2}{9a^2h^2} \tag{134}$$

$$Q_{\varphi\varphi} = \frac{7c^4}{8a^4h^2} \tag{135}$$

$$Q_{\omega\omega} = \frac{7c^4}{18a^4h^2} \tag{136}$$

The standard errors of the elements of the exterior orientation can be found from the standard error of unit weight of the image coordinates and the square roots of the weight numbers. In addition, the influence of the errors of the elements of the interior orientation must be taken into account. The standard errors of the elements of the interior orientation and the angles φ' and ω' were given previously in formulas (118) through (120). Assuming the standard errors of unit weight of the image coordinates at the camera and image calibration to be of the same order of magnitude as at the measurements for the resection in space, we find the following formulas for the standard errors of the elements of the exterior orientation under these conditions:

$$s_X = s_o \frac{h}{c} \sqrt{\frac{10}{3} + \frac{7c^4}{8a^4} + \frac{3c^2}{2a^2}} \tag{137}$$

$$s_Y = s_o \frac{h}{c} \sqrt{\frac{26}{9} + \frac{7c^4}{18a^4} + \frac{5c^2}{9a^2}} \tag{138}$$

$$s_Z = s_o \frac{h}{2} \sqrt{\frac{3}{2a^2} + \frac{1}{r'^2}} \tag{139}$$

$$s_{\bar{\varphi}} = s_o \sqrt{\frac{7c^2}{8a^4} + \frac{5c^2}{6r'^2}} \tag{140}$$

$$s_{\bar{\omega}} = s_o \sqrt{\frac{7c^2}{18a^4} + \frac{5c^2}{6r'^2}} \tag{141}$$

In these formulas,

h is the flying altitude;

c is the camera constant;

a represents the image coordinates of the control points (cf. the relations (130)); and

r' is the radius of the circle around the principal point upon which the points were located in connection with the calibration of the camera and the photographs.

Formulas (137) through (141) are applied to three different types of photographs, and the results are shown in Table IX.

If the single-point resection had been made in an ordinary photogrammetric plotting instrument, attention should have been paid to the errors of the interior orientation of the projector from the calibration. The standard errors of Table IX would therefore have become larger. Under the assumed conditions, single-point resection in space will give more accurate results than double-point resection in space. For a certain flying altitude, the wide-angle photograph gives the most accurate results. In reference 10, a study has been made of the variations of the accuracy with varying locations of the control points.

Table IX. Standard Errors of the Elements of the Exterior Orientation after Single-Point Resection in Space with Numerical Methods

Photographs	s_X	s_Y	s_Z	$s_{\bar{\varphi}}$	$s_{\bar{\omega}}$
Normal angle $c = 210$ mm $s_o = 0.004$ mm $r' = a = 80$ mm	$7.4 s_o \frac{h}{c}$ $\frac{h}{7140}$	$5.1 s_o \frac{h}{c}$ $\frac{h}{10300}$	$20.8 s_o \frac{h}{c}$ $\frac{h}{2500}$	$27250 s_o$ $1^{c}.1$	$23000 s_o$ $0^{c}.9$
Wide angle $c = 153$ mm $s_o = 0.006$ mm $r' = a = 100$ mm	$3.4 s_o \frac{h}{c}$ $\frac{h}{7500}$	$2.5 s_o \frac{h}{c}$ $\frac{h}{10200}$	$1.5 s_o \frac{h}{c}$ $\frac{h}{17000}$	$12732 s_o$ $0^{c}.8$	$10823 s_o$ $0^{c}.6$
Superwide angle $c = 88$ mm $s_o = 0.010$ mm $r' = a = 100$ mm	$2.2 s_o \frac{h}{c}$ $\frac{h}{4000}$	$1.9 s_o \frac{h}{c}$ $\frac{h}{4600}$	$0.7 s_o \frac{h}{c}$ $\frac{h}{12500}$	$7003 s_o$ $0^{c}.7$	$6366 s_o$ $0^{c}.6$

8. Tolerance Limits of the Final Results of the Photogrammetric Procedure. The standard errors of the final coordinates and elevations of terrain points have been determined previously under different assumptions concerning the fundamental operations of photogrammetry. Under such conditions, it is always of interest to estimate the maximum deviations to be expected or to be allowed if the photogrammetric data are compared with geodetic check data, which are determined with such a high geometrical quality that they can be regarded as errorless, at least in comparison with the photogrammetric quality.

Up to the present time, the "maximum errors" or tolerance limits are usually assumed to be three times the standard errors. This is a crude rule, which probably originates from the fact that in a normal distribution of errors there is a probability of only about 0.27 percent or 1:370 that an individual error exceeds three times the standard deviation of the distribution. This rule assumes a large number of observations. In practice, this rule is usually regarded to be fulfilled if the number of the observations (errors) is larger than 30. If the number is smaller than 30, the mentioned rule for estimating the maximum error or the tolerance limit becomes

approximate. Therefore, it is more correct to use procedures which have been derived for a limited number of observations for the determination of the basic accuracy (the standard error of unit weight). Such a procedure for the determination of tolerance limits of individual results of the photogrammetric (coordinates and elevations of terrain points, elements of the exterior orientation, and the like) is given by the t-test.

The distribution of the residuals (errors), from which the standard error of unit weight has been computed, must be assumed to be normal at a reasonable level (the 5 percent level is the usual one), and this assumption should have been checked in a satisfactory number of tests in connection with calibration procedures.

From the number n-u of redundant observations in the determination of the standard error of unit weight and at a certain (tolerance, fiducial) level p percent the factor t can be determined with which the standard error is to be multiplied for the determination of the tolerance limits.

The determination of the standard error of unit weight of the y-parallaxes has been made from y-parallax measurements in 6, 9, or 15 model points. This means that there are 1, 4, and 10 redundant observations, respectively. The level is usually chosen at 5 percent but 1 percent also is used. For the combinations of 1, 4, and 10 redundant observations and the levels 5 and 1 percent, the following t-values are found from a table of the t-distribution.

$$n-u = 1 \qquad t_5 = \pm 12.7 \qquad t_1 = \pm 63.7$$

$$n-u = 4 \qquad t_5 = \pm 2.8 \qquad t_1 = \pm 4.6$$

$$n-u = 10 \qquad t_5 = \pm 2.2 \qquad t_1 = \pm 3.2$$

Consequently, if the y-parallax measurements are made in 15 points for the determination of the standard error of unit weight and the 5 percent level is chosen, the tolerance limits (maximum errors) are obtained as $\pm$ 2.2s where s is the standard error of the actual result. This means that the statement that the maximum deviations between the measured (computed) value of the data (coordinates, elevations, and elements of the exterior orientation) and their true values should not exceed 2.2 times their standard errors, can be wrong in 5 percent of the tests. The statement that the maximum deviations should not exceed 3.2 times the standard error can be 1 percent wrong. There are certain approximations in this procedure, too, for the determination of the tolerance limits but they may be accepted, in particular, because the level is more or less a choice.

If root mean square values of true discrepancies have been determined, e. g., between photogrammetric and geodetic coordinates and elevations, and these root mean square values are to be compared with theoretical values as determined from the standard errors of the fundamental operations, another type of test must be used, founded upon the chi^2 distribution. Examples of such root mean square values from a theoretical derivation are given in formulas (67a), (68a), (78), (79), and (82). The tolerance limits for 4 and 10 redundant observations and at the 5 and 1 percent level are as follows:

$n-u = 4$, level 5 percent: $0.6s_0-2.9s_0$; level 1 percent: $0.5s_0-4.4s_0$

$n-u = 10$, level 5 percent: $0.7s_0-1.8s_0$; level 1 percent: $0.6s_0-2.2s_0$

This means that if the standard error of unit weight of the y-parallaxes has been determined from observations in 15 points and if the level 5 percent is chosen, the true root mean square value of elevation discrepancies of a model should be within 0.7 and 1.8 times the theoretical value as determined from the standard error of unit weight. This statement can be 5 percent wrong. Also, presently there are certain approximations in the procedure.

III. DISCUSSION

Through these investigations the error propagation from the relative and absolute orientations to the final results has been determined under well-defined conditions. In particular, the compensating effects of the absolute orientation is of great importance and has theoretically to be considered for each actual combination of control points in planimetry and elevation. It should also be emphasized that the derivations have been made under the assumptions of moderate elevation differences on the ground. The effects of such differences, however, is dependent upon the actual situation concerning the relative orientation and is reduced if the residual y-parallaxes after a preliminary orientation are sufficiently small, of the order of magnitude some hundredth of a millimeter in the photographs. It should also be emphasized again that the most important regular errors of the photographs and of the instrument are assumed to be corrected for and that the residual errors are normally distributed at a reasonable level. "Critical surfaces" must be noticed, however.

The derived formula systems for the geometrical quality of the final results are of great significance for a number of problems, in particular, concerning the planning of aerial photography for plotting with defined tolerance limits. In a separate investigation, the application of the formulas to the problem of topographic mapping

will be shown. The quality determination of the elements of the exterior orientation is of interest for the test of other procedures for the determination of the exterior orientation in connection with aerial photography.

It can be mentioned that preliminary formula systems for the determination of the final geometrical quality of coordinates and elevations from photogrammetric models and derived by the author were tested in practical experiments arranged by the International Society of Photogrammetry and presented at the Stockholm and London Congresses in 1956 and 1960, respectively. At these times, specialized conditions were present and the formula systems had to be derived accordingly. The results of the comparison between the theoretical predictions with respect to the measured y-parallaxes and the practical results from the check points on the ground were entirely satisfactory. The deviations between the theoretical and practical results were always found to be within the confidence limits, determined from ordinary statistical procedures (reference 5). Therefore, it must be expected that the formula systems which have been derived and presented in this investigation according to the same basic principals are sufficiently reliable for the purpose. It is, however, desirable that further tests be made between the theoretical and the practical accuracy. For the elements of the exterior orientation such tests cannot be arranged because the photogrammetric method for the determination of the elements may be the most reliable. But, because the same principles have been used for the final model coordinates and elevations, on the one hand, and the elements of the exterior orientation, on the other, it seems probable that the reliability of the derived data for the orientation elements is satisfactory within the confidence limits which have been computed and shown.

Finally, it should be mentioned that the basic information on the geometrical quality is obtained from the contents of the photographs only, and that in principle no additional control data are assumed to be available. On the other hand, redundant control is always welcome in practice for checks against gross errors and of the theoretical geometrical quality.

IV. CONCLUSIONS

9. Conclusions. On the basis of the results of the investigation reported herein, it is concluded that:

a. The method of least squares and its law of error propagation allow a well-defined theoretical determination of the geometrical quality to be expected in the final results of the photogrammetric procedure in terms of the standard errors of the y-parallaxes.

b. The theoretical geometrical quality of the elements of the exterior orientation after double- and single-point resections in space can be determined in this way only.

c. The influence of the errors of the interior orientation must be taken into account. A well-defined procedure for the calibration of the camera and photographs is therefore necessary. Also, here the method of least squares is of basic importance.

d. Tolerance limits for the relative and absolute orientations can be derived from the results of these investigations.

REFERENCES

1. Hallert, B., Investigations into the Basic Geometrical Quality of Aerial Photographs. Technical Paper. GIMRADA, Fort Belvoir, Virginia, 1962.

2. __________, Tests of Basic Geometrical Qualities of Photogrammetric Plotting Instruments. Technical Paper. GIMRADA, Fort Belvoir, Virginia, 1962.

3. __________, "Contribution to Theory of Errors for Double Point Intersection in Space," Transactions of The Royal Institute of Technology, Stockholm, Sweden. Nr. 35, 1950.

4. __________, "Über die Genauigkeit dër Luftphotogrammetrie," Transactions of The Royal Institute of Technology, Stockholm, Sweden. Nr. 99, 1956.

5. Hallert, B., Ottoson, L., and Ternryd, C. O., Fundamental Questions in Relation to Controlled Experiments. Report from Subcommission IV:4, I. S. P. International Congress for Photogrammetry, London, 1960.

6. Hallert, B., Photogrammetry. McGraw-Hill, New York, N. Y., 1960.

7. Hallert, B. and Ottoson, L., "General Differential Formulae of the Complete Projective Relations between Planes," Transactions of The Royal Institute of Technology, Stockholm, Sweden. Nr. 140, 1959.

8. Jonsson, Ake, "On the Accuracy of Stereoplotting of Convergent Aerial Photographs," Photogrammetria. Nr. 3, 1960-1961.

9. Ottoson, L., "Determination of the Accuracy of Double and Single Point Resection in Space," Division of Photogrammetry, The Royal Institute of Technology, Stockholm, Sweden, 1957.

10. Welander, E., "Some Aspects of the Accuracy of Single Point Resection in Space," Division of Photogrammetry, The Royal Institute of Technology, Stockholm, Sweden, 1958.

Correction Sheet to GIMRADA Research Note No. 6

"Investigation of the Geometrical Quality of the Relative and Absolute Orientation Procedures and the Final Results of the Photogrammetric Procedure"

Page 4	expression (5)	Read: $+ \frac{h^2 + (x-b)^2}{b} d\varphi_2$
Page 14	expression (36)	Read: $s_o = \frac{\sqrt{[vv]}}{2}$
Page 15	expression (43)	Read: $s_o = \frac{\sqrt{[vv]}}{2}$
Page 27	line 19	Read: a prime sign in order.....
Page 29	expression (62)	Read: $+ s_o^2 \frac{x^2 h^2}{b^2 d^2} \left\{ \frac{(x-b)^2}{b^2} + \frac{3y^2}{4d^2} \right\}$
Page 33	expression (75)	Read: $- \left\{ \frac{(x-b)^2}{b} + \frac{x-b}{2} \right\} \frac{y}{h} d\varphi_2$
Page 51	expression (122)	Read: $\left(\frac{1}{3} + \frac{7b^2}{6h^2} + \frac{b^2}{2d^2} + \frac{h^2}{2d^2} + \frac{2h^2}{3b^2} + \frac{3b^2h^2}{4d^4} \right)$

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